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CHAPTER 2

INTRO. TO INTEGERS

We are at the very fortunate stage now, where we can begin to reap the benefits of a reasonable understanding of the basic axioms for integers. The flavor of the course will now change. Rather than learning more axioms or rules, we will take the ones we have learned and see them as building blocks, which we will use to construct logical bits of poetry which we will call *theorems*.

2.1 Multiplying by Zero

Recall that our historical affair with numbers. First there was nothing by the natural numbers.

$$\mathbb{N} = \{1, 2, 3, 4....\}$$

We then invented the round number, the one they call 0. This increased our set, and in doing so, we renamed this slightly larger set to be the set of whole numbers.

$$\mathbb{W} = \{0, 1, 2, 3, 4....\}$$

Now let us pause here and imagine the commotion, the rumors, and the gossip that spread across the lands when this new number was invented. "What is this new round number with a hole people speak of?, and what does it do?", people whispered. Of course, it was invented to be the Additive identity element, causing no change to any number when added to it. Then people started wondering, "can one multiply by this round thing they call zero?" and if so "how?".

Thus people wondered about simple questions such as how much is

$$4 \cdot 0$$

Of course, we can not use the to resolve this persay as 0 is not a natural number. However, the answered lies on the axioms. We can come to the suspected answer, $4 \cdot 0 = 0$, with just a bit of patience and creativity. We shall proceed using only our audacity and the established rules. Observe;

Example

$$0 + 0 = 0 \qquad (\quad)$$

$$4(0 + 0) = 4(0) \qquad (\quad)$$

$$4 \cdot 0 + 4 \cdot 0 = 4 \cdot 0 \qquad (\quad)$$

note the first line is creative in the sense that we start with $0+0 = 0$, which while clearly true, to the unversed logical poet may seem totally unrelated to the sought statement $4 \cdot 0 = 0$. But wait there is more, once we have slapped that 4 on each side and we have distributed as noted, we begin to sense there

Example

$$0+0=0 \qquad (\quad)$$

$$5(0+0)=5(0) \qquad (\quad)$$

$$5\cdot 0+5\cdot 0=5\cdot 0 \qquad (\quad)$$

$$(5\cdot 0+5\cdot 0)+-5\cdot 0=(5\cdot 0)+-5\cdot 0 \qquad (\quad)$$

$$5\cdot 0+(5\cdot 0+-5\cdot 0)=5\cdot 0+-5\cdot 0 \qquad (\quad)$$

$$5\cdot 0+0=0 \qquad (\quad)$$

$$5\cdot 0=0 \qquad (\quad)$$

One can then naturally ask about $6\cdot 0$, $7\cdot 0$, or even $\frac{1}{5}\cdot 0$, but soon enough the humanness and our 3lb brain begin to take over. We observe that there really would not be much change in the line of thinking as outline in the last couple of examples. One begins to see that pattern leading to a very convincing proposition, namely that any number, a, so long as it obeys our axiom must inevitably fall victim to the 0 as in $a\cdot 0=0$ and if that is true

by it must follow that $0\cdot a=0$. Said differently, instead of proving that $4\cdot 0=0$ and $5\cdot 0$, etc etc. one by one, we can use "a" as a template:

Theorem 2.1.1. *Zero Multiplication Theorem [0MT] Assume a represents a number or symbol for which the [chapter one] axioms apply. Then*

$$a\cdot 0=0$$

and $0\cdot a=0$

is a relation ship. Consider the last equation above and specifically consider what would happen if we "kill' a "$4\cdot 0$" from each side? Talk is cheap, let us do what needs to be done.

$$0+0=0 \qquad (\quad)$$

$$4(0+0)=4(0) \qquad (\quad)$$

$$4\cdot 0+4\cdot 0=4\cdot 0 \qquad (\quad)$$

$$(4\cdot 0+4\cdot 0)+-4\cdot 0=(4\cdot 0)+-4\cdot 0 \qquad (\quad)$$

$$4\cdot 0+(4\cdot 0+-4\cdot 0)=4\cdot 0+-4\cdot 0 \qquad (\quad)$$

$$4\cdot 0+0=0 \qquad (\quad)$$

$$4\cdot 0=0 \qquad (\quad)$$

Now consider the next logical question. Considering that $4\cdot 0=0$ has been established, can we no allow ourselves to wonder how much would $5\cdot 0$ turn out to be?.

Proof.

$$0 + 0 = 0$$

$$a(0 + 0) = a(0)$$

$$a \cdot 0 + a \cdot 0 = a \cdot 0$$

$$(a \cdot 0 + a \cdot 0) + -a \cdot 0 = (a \cdot 0) + -a \cdot 0$$

$$a \cdot 0 + (a \cdot 0 + -a \cdot 0) = a \cdot 0 + -a \cdot 0$$

$$a \cdot 0 + 0 = 0$$

$$a \cdot 0 = 0$$

$$0 \cdot a = 0$$

∎

By proving the template, we intend to provide a proof for all possible values of a[1], thus at once proving it for *all* of our number, so long as they play nice with our axioms. Another way to think about theorems is to seem them as a carefully cooked up sequence of repeatable steps that are *packaged* into one package containing all the repetable steps inside the package. Once we have proven *0MT* we own it, our forever, and can not be stolen or take away. We can now use it as such:

Example

$$5 \cdot 0 = 0 \tag{0mt}$$

We can 0MT all night long:

[1] so long as a obeys our chapter one axioms

Examples

$$7 \cdot 0 = 0 \tag{0MT}$$

$$13 \cdot 0 = 0 \tag{0MT}$$

$$x \cdot 0 = 0 \tag{0MT}$$

$$0 \cdot \frac{3}{5} = 0 \tag{0MT}$$

$$0 \cdot x^4 = 0 \tag{0MT}$$

$$0 \cdot (-1) = 0 \tag{0MT}$$

2.1.1 Exercises

1. Using only our axioms from chapter 1, prove that $-1 \cdot 48 = -48$

2. Using only our axioms from chapter 1, prove that $-1 \cdot 8 = -8$

3. Using only our axioms from chapter 1, prove that $-1 \cdot 50 = -50$

2.2 Multiplying Integers

In last section, we focused completely on the idea of multiplying by zero, resulting in the famous [0MT]. At this point should we comfortable multiplying by any of the numbers in the set of *whole numbers*.

$$\mathbb{W} = \{0, 1, 2, 3, 4. \dots\}$$

Now we turn out attention to multiplying by *integer numbers*:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4. \dots\}$$

In other words, we would like to learn how to multiply by $-1, -2, -3$, etc. etc. Let us first begin by learning how to multiply by -1. Consider for example how much is $-1 \cdot 5$? Again, we will summon the axioms to provide a more than graceful solution. Carefully consider the following.:

Example

$$\begin{aligned}
-1 + 1 &= 0 && \text{(Ainv)}\\
(-1 + 1)5 &= (0)5 && \text{(CLM)}\\
-1 \cdot 5 + 1 \cdot 5 &= 0 \cdot 5 && \text{(DL)}\\
-1 \cdot 5 + 1 \cdot 5 &= 0 && \text{(0MT)}\\
-1 \cdot 5 + 5 &= 0 && \text{(Mid)}\\
(-1 \cdot 5 + 5) + -5 &= 0 + -5 && \text{(CLA)}\\
-1 \cdot 5 + (5 + -5) &= 0 + -5 && \text{(ALA)}\\
-1 \cdot 5 + 0 &= 0 + -5 && \text{(AinV)}\\
-1 \cdot 5 &= -5 && \text{(Aid)}
\end{aligned}$$

Notice what the eminent conclusion: $-1 \cdot 5 = -5$, one way to read this is to say "negative one times 5 is equal to negative 5", yet a different way to think about this is to see it as "the additive inverse of 1 times 5 is equal to the additive inverse of 5". The second way may be a deeper more powerful way of looking at it, as the proof provided in the example above makes heave use of the fact that -1 is, by definition "the additive inverse of 1" and likewise for -5. Here's another example.

Example

$$\begin{aligned}
-1 + 1 &= 0 && \text{(Ainv)}\\
(-1 + 1)7 &= (0)7 && \text{(CLM)}\\
-1 \cdot 7 + 1 \cdot 7 &= 0 \cdot 7 && \text{(DL)}\\
-1 \cdot 7 + 1 \cdot 7 &= 0 && \text{(0MT)}\\
-1 \cdot 7 + 7 &= 0 && \text{(Mid)}\\
(-1 \cdot 7 + 7) + -7 &= 0 + -7 && \text{(CLA)}\\
-1 \cdot 7 + (7 + -7) &= 0 + -7 && \text{(ALA)}\\
-1 \cdot 7 + 0 &= 0 + -7 && \text{(AinV)}\\
-1 \cdot 7 &= -7 && \text{(Aid)}
\end{aligned}$$

Of course what we are really after is documenting a pattern or a generic template that shows the steps always work not just for 5 or 7 but for a generic template variable such as a. We do so and package all the steps into one theorem, which we will use freely from here on.

Theorem 2.2.1. *Minus Theorem [MT] Assume a represents a number or symbol for which the [chapter one] axioms apply. Then*

$$-1 \cdot a = -a$$

and

$$a \cdot -1 = -a$$

Proof.

$$\begin{aligned}
-1 + 1 &= 0 && \text{(Ainv)}\\
(-1 + 1)a &= (0)a && \text{(CLM)}\\
-1 \cdot a + 1 \cdot a &= 0 \cdot a && \text{(DL)}\\
-1 \cdot a + 1 \cdot a &= 0 && \text{(0MT)}\\
-1 \cdot a + a &= 0 && \text{(Mid)}\\
(-1 \cdot a + a) + -a &= 0 + -a && \text{(CLA)}\\
-1 \cdot a + (a + -a) &= 0 + -a && \text{(ALA)}\\
-1 \cdot a + 0 &= 0 + -a && \text{(AinV)}\\
-1 \cdot a &= -a && \text{(Aid)}\\
a \cdot -1 &= -a && \text{(CoLA)}
\end{aligned}$$

Notice that read from left to right MT says how one multiplies by a -1,

$$-1 \cdot 6 = -6 \qquad [\text{MT}]$$

but, when read from right to left, it says that on any number that has a "-" infront of it one can stick a 1 between the "-" and the number such as

$$-6 = -1 \cdot 6 \qquad [\text{MT}]$$

or

$$-B = -1 \cdot B \qquad [\text{MT}]$$

or even

$$-(x+1) = -1 \cdot (x+1) \qquad [\text{MT}]$$

The next logical step is to learn to multiply by other negative numbers, such as $-4 \cdot 5$, and now we see the theorems begin to pile up. Observe how by learning new theorems, we become more powerful and graceful in our ways. For example, the next idea show that once we have learned to multiply by -1 and with the use MT we have done most of the work already.

Example

$$\begin{aligned} & -4 \cdot 5 && \text{(given)} \\ =& (-1 \cdot 4) \cdot 5 && \text{(MT)} \\ =& -1 \cdot (4 \cdot 5) && \text{(ALM)} \\ =& -1 \cdot 20 && \text{(TT)} \\ =& -20 && \text{(MT)} \end{aligned}$$

Indeed what this example shows that a negative 4 times positive 5 equals a negative twenty. By looking at the steps in the proof one can generalize and realize it would be try of any numbers $-a$ and b, their product would be $-ab$. We will prove and package this theorem below, adding power to our arsenal and repertoire of tight and logically graceful theorems.

Theorem 2.2.2. *Negative Times Positive Theorem [NPT] If $-a$ and b represent number or symbols for which our axioms apply then:*

$$-a \cdot b = -ab$$

Proof.

$$\begin{aligned} & -a \cdot b && \text{(given)} \\ =& (-1 \cdot a) \cdot b && \text{(MT)} \\ =& -1 \cdot (ab) && \text{(ALM)} \\ =& -ab && \text{(MT)} \end{aligned}$$

∎

Note when using NPT theorem on variables we will use it just as above, when using it with integers or other numbers we will include TT ot other number-multiplying skills to carry out the last step. For example rather than saying

$$-4 \cdot 5 = -(4 \cdot 5)$$

as above we will include the TT step to say

$$-4 \cdot 5 = -20 \qquad \text{NPT}$$

But wait there is more! Since we are having a theorem fest here let us take it a couple steps further. Let us consider multiplying a negative times another negative!.. such as $-1 \cdot -1$ This one is so special we will skip the examples and simply jump to the packaging of it into a special theorem.

Theorem 2.2.3. *Negative One Times Negative One Theorem [NotNot]*

$$-1 \cdot -1 = 1$$

Proof.

$$-1+1=0$$

$$(-1+1)\cdot -1 = 0\cdot -1$$

$$-1\cdot -1 \quad + \quad 1\cdot -1 = 0\cdot -1$$

$$-1\cdot -1 \quad + \quad 1\cdot -1 = 0 \qquad \text{0MT}$$

$$-1\cdot -1 \quad + \quad -1 = 0$$

$$(-1\cdot -1 + -1)+1 = 0+1$$

$$-1\cdot -1 + (-1+1) = 0+1$$

$$-1\cdot -1 + 0 = 0+1$$

$$-1\cdot -1 = 1$$

∎

But wait there is more! Our next theorem goes a step beyond the NoTNoT Theorem. While the NoTNoT theorem speaks only of -1 and -1, the next theorem generalizes for any negative number times any negative number. It states that the product of a negative number times a negative numbers will be a positive number . Symbolically:

$$(-x)(-y) = xy \qquad [NNT]$$

Theorem 2.2.4. *Negative Times Negative Theorem [NNT]*

$$-x\cdot -y = xy$$

The reader is invited to prove this theorem. We conclude this section with some basic arithmetic example using positive and negative integers. Throughout we make frequent use of the theorems above.

Example Compute and simplify:

$$(-1)(-1)(-1) \quad + \quad 3$$

solution:

$$\begin{aligned}
(-1)(-1)(-1) \quad + \quad 3 &= (-1)\,[(-1)(-1)] \quad + \quad 3 && \text{(ALM)}\\
&= (-1)\,[1] \quad + \quad 3 && \text{(NoTNoT)}\\
&= -1 \quad + \quad 3 && \text{(MiD)}\\
&= -1 \quad + \quad (1+2) && \text{(AT)}\\
&= (-1 \quad + \quad 1) \quad + \quad 2 && \text{(ALA)}\\
&= 0 \quad + \quad 2 && \text{(AINV)}\\
&= 2 && \text{(AId)}
\end{aligned}$$

2.2.1 Exercises

1. Using only 0MT and/or all prior axioms/theorems prove that

$$-1\cdot 33 = -33$$

2. Using only 0MT and/or all prior axioms/theorems prove that

$$-1\cdot 31 = -31$$

3. Using only 0MT and/or all prior axioms/theorems prove that

$$-1\cdot 22 = -22$$

4. Using only 0MT and/or all prior axioms/theorems prove

$$-1\cdot u = -u$$

5. Using only 0MT and/or all prior axioms/theorems prove

$$-1\cdot w = -w$$

6. Using only MT and/or all prior axioms/theorems prove

$$-6\cdot 2 = -12$$

7. Using only MT and/or all prior axioms/theorems prove

$$-13 \cdot 4 = -52$$

8. Using only MT and/or all prior axioms/theorems prove

$$-8 \cdot 7 = -56$$

9. Using only NoTNoT and/or all prior axioms/theorems prove

$$-13 \cdot -5 = 65$$

10. Using only NoTNoT and/or all prior axioms/theorems prove

$$-6 \cdot -2 = 12$$

11. Using only NoTNoT and/or all prior axioms/theorems prove

$$-13 \cdot -7 = 91$$

2.3 Adding Integers

Here lies another milestone. We are now ready to learn how to learn to add with negative numbers, more generally, to adding *integer numbers*:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$$

In other words, we would like to learn how to add by $-1, -2, -3$, etc. etc. Consider for example how much is $-3 + -6$? Again, we will summon the axioms and all the theorems, now in now under our belt, to resolve this. Carefully consider the following.:

Example Add $-3 + -6$.
solution:

$$\begin{aligned} & -3 + -6 && \text{(given)} \\ = & -1 \cdot 3 + -1 \cdot 6 && \text{(MT)} \\ = & -1(3+6) && \text{(DL)} \\ = & -1 \cdot 9 && \text{(AT)} \\ = & -9 && \text{(MT)} \end{aligned}$$

This example points to a very important idea. Namely, that two negative numbers when added become a 'larger but still negative' number. Moreover, this appears to be a good moment to point out the richness of the ideas we are developing and how these relate to our very own human dignity. The process outlined on the above example is not the only way to do it, and to advice you to always do it that way would be a complete and utter disregard for your creativity and innate human cognitive brilliance. Rather than treating you like an insect, we opt for a more dignified approach. We encourage you to unleash your creativity, and be confident in the rules and theorems we have learned and your 3 lb. brain, as these will make you a composer of thought.

For example, read the following alternative approach to the same question, then try your own way, any way your mind fancies. The only condition is that you observe the rules and theorems we have learned. A good way to see the axioms and theorems is to see them as ingredients in your very own gourmet kitchen, and you are the human chef with complete dignity and freedom.

Example Add $-3 + -6$.

solution: This time we resolve it based on the idea that we might want to kill the -3 and the -6, which to do so we need a positive 3 and a positive 6 , thus we need a total of 9 to kill these... as in...

$$\begin{aligned}
&-3+-6 && \text{(given)}\\
=&0+-3+-6 && \text{(AiD)}\\
=&(-9+9)+-3+-6 && \text{(AInv)}\\
=&-9+9+-3+-6 && \text{(ALA)}\\
=&-9+(3+6)+-3+-6 && \text{(AT)}\\
=&-9+(3+-3)+(6+-6) && \text{(ALA, CoLA)}\\
=&-9+0+0 && \text{(AinV)}\\
=&-9 && \text{(AiD)}
\end{aligned}$$

As usual, this seems like something we can get a lot of mileage from, so we shall formally make it and prove it as a theorem, adding to our already dangerously powerful collection of tools.

Theorem 2.3.1. *Negative + Negative Theorem [N+NT] Assume a and b represents a numbers or symbols for which the [chapter one] axioms apply. Then*

$$-a+-b=-(a+b)$$

Proof.

$$\begin{aligned}
&-a+-b && \text{(given)}\\
=&-1\cdot a+-1\cdot b && \text{(MT)}\\
=&-1(a+b) && \text{(DL)}
\end{aligned}$$

note when a and b are natural numbers we will include the AT in the last step.

A few more words are in order with respect to the N+NT. When read backwards it becomes a nice tool to add numbers of mixed sign. For example, from left to right we may use the N+NT to say something like

$$-3+-5=-8$$

but it we look at it in the other direction the N+NT would say..

$$-8=-3+-5$$

The following example shows at least one way this might be used..

Example Add $-8+5$.
solution:

$$\begin{aligned}
&-8+5 && \text{(given)}\\
=&(-3+-5)+5 && \text{(N+NT)}\\
=&-3+(-5+5) && \text{(ALA)}\\
=&-3+0 && \text{(AinV)}\\
=&-3 && \text{(AiD)}
\end{aligned}$$

Example Add $-5+3$.
solution:

$$\begin{aligned}
&-5+3 && \\
=&(-2+-3)+3 && \text{(N+NT)}\\
=&-2+(-3+3) && \text{(ALA)}\\
=&-2+(0) && \text{(AInV)}\\
=&-2 && \text{(AiD)}
\end{aligned}$$

Example Compute and simplify:

$$(-1)(-1)(-1)+3$$

solution:

$$\begin{aligned}
(-1)(-1)(-1)+3 &= (-1)\,[(-1)(-1)]+3 && \text{(ALM)}\\
&= (-1)\,[1]+3 && \text{(NoTNoT)}\\
&= -1+3 && \text{(MiD)}\\
&= -1+(1+2) && \text{(AT)}\\
&= (-1+1)+2 && \text{(ALA)}\\
&= 0+2 && \text{(AINV)}\\
&= 2 && \text{(AId)}
\end{aligned}$$

2.3.1 Exercises

1. carefully simplify the following:

$$-14+8$$

2. carefully simplify the following:

$$-10+3$$

3. carefully simplify the following:

$$-23+11$$

4. carefully simplify the following:

$$2+-3$$

5. carefully simplify the following:

$$13+-16$$

6. carefully simplify the following:

$$12+-23$$

7. (proving N+NT) Using only prior axioms and theorems prove

$$-26=-12+-14$$

8. (proving N+NT) Using only prior axioms and theorems prove

$$-8=-5+-3$$

9. (proving N+NT) Using only prior axioms and theorems prove

$$-20=-13+-7$$

10. (proving N+NT) Using only prior axioms and theorems prove

$$-c+-x=-(c+x)$$

11. (proving N+NT) Using only prior axioms and theorems prove

$$-c+-x=-(c+x)$$

12. Using only prior axioms and theorems up to N+NT to simplify

$$2+8(0+5)+-4$$

13. Using only prior axioms and theorems up to N+NT to simplify

$$-7+-1(5+2)+-2$$

14. Using only prior axioms and theorems up to N+NT to simplify

$$-9+-10(3+3)+4$$

15. Using only prior axioms and theorems up to N+NT to simplify

$$10+-6(2+-1)+1$$

16. Using only prior axioms and theorems up to N+NT to simplify

$$-5+0(-1+-3)+4$$

2.4 Subtraction

Thus far, all our attention has been on addition and or multiplication, and for good reason. Almost all of our axioms and theorems play well with these two operations, addition and with multiplication. That is much more than we can say for subtraction or division. In fact, these usually do not play well with our axioms. However, in this section we introduce a powerful way of defining subtraction, effectively changing all subtraction statements into addition statements. By changing all subtraction to some form of addition we effectively turn these expression into something that plays nicely with all of our axioms and all of our theorems thus far.

Definition *Subtraction [def a-b]* will be defined as a binary operation with the symbol "-", and it will be defined as

$$a - b = a + -b$$

said differently, 'to subtract b" will mean "to add the additive inverse of b"

Example Subtract $10 - 6$.
solution:

$$\begin{aligned} & 10 - 6 && \text{(given)} \\ & 10 + -6 && \text{(def a-b)} \\ =& (4+6) + -6 && \text{(AT)} \\ =& 4 + (6 + -6) && \text{(ALA)} \\ =& 4 + 0 && \text{(AinV)} \\ =& 4 && \text{(AinV)} \end{aligned}$$

Example Subtract $-3 - 6$.
solution:

$$\begin{aligned} & -3 - 6 && \text{(given)} \\ & -3 + -6 && \text{(def a-b)} \\ =& -1 \cdot 3 + -1 \cdot 6 && \text{(MT)} \\ =& -1(3+6) && \text{(DL)} \\ =& -1 \cdot 9 && \text{(AT)} \\ =& -9 && \text{(MT)} \end{aligned}$$

Of course, these are only a couple approaches to the problem. There are infinite many ways of doing it and you are encouraged in composing your very own method, anything your mind fancies is acceptable, so long as you mind the rules and theorems we have learned. Here is one more example.

Example Subtract $-3 - (-6) - 2(1+3)$.
solution:

$$\begin{aligned} & -3 - (-6) - 2(1+3) && \text{(given)} \\ =& -3 + -(-6) + -2(1+3) && \text{(def a-b)} \\ =& -3 + -1(-6) + -2(1+3) && \text{(MT)} \\ =& -3 + 6 + -2(1+3) && \text{(NNT)} \\ =& -3 + 6 + -2(4) && \text{(AT)} \\ =& -3 + 6 + -8 && \text{(NPT)} \\ =& (-3 + -8) + 6 && \text{(ALA, CoLA)} \\ =& -11 + 6 && \text{(N+NT)} \\ =& (-5 + -6) + 6 && \text{(N+NT)} \\ =& -5 && \text{(ALA, AinV, AID)} \end{aligned}$$

2.4.1 Exercises

1. Using only prior axioms and theorems to simplify:

$$3 - 11 - 29$$

2. Using only prior axioms and theorems to simplify:

$$5 - 10 - 7$$

3. Using only prior axioms and theorems to simplify:

$$-7 - 6 - 7$$

4. Using only prior axioms and theorems to simplify:

$$2 - 1 - 22$$

5. Using only prior axioms and theorems to simplify:

$$19 - 13(5 - 6) - 8$$

6. Using only prior axioms and theorems to simplify:

$$20 - 3(3 - 4) - 21$$

7. Using only prior axioms and theorems to simplify:

$$14 - 10(1 - 2) - 14$$

8. Using only prior axioms and theorems to simplify:

$$16 - 1(29 - 6) - 23$$

9. Using only prior axioms and theorems to simplify:

$$25 - 9(24 - 2) - 22$$

10. Using only prior axioms and theorems to simplify:

$$27 - 12(23 - 3) - 6$$

11. Using only prior axioms and theorems to simplify:

$$-9 - 2(17 - 7) - 0$$

12. Using only prior axioms and theorems to simplify:

$$7 - 9(20 - 2) - -2$$

13. Using only prior axioms and theorems to simplify:

$$-8 - 9(23 - 15) - 5$$

14. Using only prior axioms and theorems to simplify:

$$-4 - 7(14 - 12) - -2$$

2.5 Natural Exponents

Over the years, we have developed a very nice and concise way to write the product of a number times itself several times. For example, rather than writing

$$5 \cdot 5 \cdot 5 \cdot 5$$

We will adopt the following convention: we will write 5^4 to represent exactly the same quantity as $5 \cdot 5 \cdot 5 \cdot 5$. The 4 is usually called the '*exponent*' and the 5 is usually called the *base*. For now, and from now on, we will adopt this meaning every time we encounter a small natural number next to the upper, right-side corner of another number. Conversely, every time we encounter a product of a bunch of '5's, for example, we can opt to rewrite it as just one 5 with an exponent equal to the number of times it is being multiplied. We formally state the definition for natural exponents here:

Definition *Natural Exponents [$\mathbb{N}$-Expo]* Assume n is a natural number, and A is any number, variable, or symbol/s for which our axioms apply. Then we define A^n as:

$$A^n = \underbrace{A \cdot A \cdot A \cdot \cdots \cdot A}_{n\text{-times}}$$

Example Compute 3^4.
solution:

$$\begin{aligned} & 3^4 && \text{(given)} \\ & = 3 \cdot 3 \cdot 3 \cdot 3 && \text{(N-Expo)} \\ & = (3 \cdot 3)(3 \cdot 3) && \text{(ALM)} \\ & = (9)(9) && \text{(TT)} \\ & = 81 && \text{(TT)} \end{aligned}$$

Here are some more examples.

Example Expand $(3x)^2$
solution:

$$\begin{aligned} &(3x)^2 && \text{(given)} \\ &= (3x)(3x) && \text{(N-Expo)} \end{aligned}$$

The point is worth repeating, again! Look carefully at...

Example Expand $-x^2$
solution:

$$\begin{aligned} &-x^2 && \text{(given)} \\ &= -xx && \text{(N-Expo)} \end{aligned}$$

Example Expand $(-x)^2$
solution:

$$\begin{aligned} &(-x)^2 && \text{(given)} \\ &= (-x)(-x) && \text{(N-Expo)} \end{aligned}$$

Now, consider what happens when we multiply numbers of the form

$$3^2 \cdot 3^4$$

Using the N-Expo definition we obtain:

$$\begin{aligned} &3^2 \cdot 3^4 && \\ &= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 && \text{(N-Expo)} \\ &= 3^6 && \text{(N-Expo)} \end{aligned}$$

There is clearly a pattern here that we can observe. When we multiply two numbers with the same base, the result is that same base raised to the sum of the exponents. Thus we have the very famous *Just Add the Exponents [JAE]* theorem. Formally,

Theorem 2.5.1. *Just Add Exponents [JAE] Suppose n and m represent natural numbers, and A represents a number for which our axioms apply. Then*

$$A^n \cdot A^m = A^{n+m}$$

Examples

$$\begin{aligned} 5^1 &= 5 && \text{N-Expo} \\ 5^3 &= 5 \cdot 5 \cdot 5 && \text{N-Expo} \\ (-3)^3 &= (-3) \cdot (-3) \cdot (-3) && \text{N-Expo} \\ (xy^3)^2 &= (xy^3) \cdot (xy^3) && \text{N-Expo} \\ xy^3 &= xyyy && \text{N-Expo} \\ -2^3 &= -2 \cdot 2 \cdot 2 && \text{N-Expo} \\ \left(\frac{x}{2y^2}\right)^3 &= \left(\frac{x}{2y^2}\right) \cdot \left(\frac{x}{2y^2}\right) \cdot \left(\frac{x}{2y^2}\right) && \text{N-Expo} \end{aligned}$$

It is incredibly important to note that by convention we define the base to be only whatever is next to the exponent. Consider the distinction between the following two examples.

Example Expand abc^4.
solution:

$$\begin{aligned} &abc^4 && \text{(given)} \\ &= abcccc && \text{(N-Expo)} \end{aligned}$$

Example Expand $(abc)^4$.
solution:

$$\begin{aligned} &(abc)^4 && \text{(given)} \\ &= (abc)(abc)(abc)(abc) && \text{(N-Expo)} \end{aligned}$$

These are indeed different quantities on most occasions. The point is worth repeating.

Example Expand $3x^2$
solution:

$$\begin{aligned} &3x^2 && \text{(given)} \\ &= 3xx && \text{(N-Expo)} \end{aligned}$$

Example

$$\begin{aligned} &4^2 \cdot 4^5 \\ &= 4^{2+5} && \text{(JAE)} \\ &= 4^7 && \text{(AT)} \end{aligned}$$

Example

$$\begin{aligned} &4^3 \cdot 4^2 \\ &= 4^{3+2} && \text{(JAE)} \\ &= 4^5 && \text{(AT)} \end{aligned}$$

Example

$$\begin{aligned} &9^7 \cdot 4^3 \\ &= 9^{10} && \text{(JAE, AT)} \end{aligned}$$

Example

$$\begin{aligned} &4^8 \\ &= 4^{5+3} && \text{(AT)} \\ &= 4^5 \cdot 4^3 && \text{(JAE)} \end{aligned}$$

Example

$$\begin{aligned} &4^8 \\ &= 4^{2+6} && \text{(AT)} \\ &= 4^2 \cdot 4^6 && \text{(JAE)} \end{aligned}$$

Exponents, are often called 'powers'. For example, 3^5 may be read as 3 to the power of 5, or 3 to the 5th power. Next, we consider the possibility of an expression that contains a 'powers of powers'. For example consider the meaning of

$$\left(3^4\right)^2$$

It turns out, we can resolve this by simply using the definition of N-Expo, as in:

Example

$$\begin{aligned} \left(3^4\right)^2 &= \left(3^4\right)\left(3^4\right) && \text{(N-Expo)} \\ &= 3^{4+4} && \text{(JAE)} \\ &= 3^8 && \text{(AT)} \end{aligned}$$

There is clearly a nice pattern here to observe. Observe we shall, and we will make a theorem out of it.

Theorem 2.5.2. *Power to Power [P2P] Suppose n and m represent natural numbers, and A represents a number for which our axioms apply. Then*

$$(A^n)^m = A^{nm}$$

Moreover, for several bases, A, B, & C, and several powers we have:

$$\left(A^n B^k C^p\right)^m = A^{nm} B^{km} C^{pm}$$

Using P2P, the previous example may look something like this:

Example

$$\begin{aligned} &\left(3^4\right)^2 \\ &= 3^{2\cdot 4} && \text{(P2P)} \\ &= 3^8 && \text{(TT)} \end{aligned}$$

Example

$$\begin{aligned} &\left(4^3\right)^2 \\ &= 4^6 && \text{(P2P)} \end{aligned}$$

Example

$$\begin{aligned} &\left(4^7\right)^2 \\ &= 4^{7\cdot 2} && \text{(P2P)} \\ &= 4^{14} && \text{(TT)} \end{aligned}$$

P2P -backwards!

$$\begin{aligned} &5^{12} \\ &= 5^{3\cdot 4} && \text{(TT)} \\ &= \left(5^3\right)4 && \text{(P2P)} \end{aligned}$$

In P2P goes well with the product of several bases, as in:

Example

$$\begin{aligned} &\left(4^7 \cdot 2^3\right)^2 \\ &= 4^{7\cdot 2} \cdot 2^{3\cdot 2} && \text{(P2P)} \\ &= 4^{14} \cdot 2^6 && \text{(TT)} \end{aligned}$$

Example

$$\begin{aligned} &\left(4^7 x^3 y^5\right)^2 \\ &= 4^{7\cdot 2} \cdot x^{3\cdot 2} \cdot y^{5\cdot 2} && \text{(P2P)} \\ &= 4^{14} x^6 y^{10} && \text{(TT)} \end{aligned}$$

2.5.1 Exercises

1. Simplify sing only prior axioms, theorems, and definitions:

$$0^3 - 16(-4-5) + -2(9)$$

2. Simplify sing only prior axioms, theorems, and definitions:

$$-2^4 + 9(4+12) - 2(12)$$

3. Simplify sing only prior axioms, theorems, and definitions:

$$4^4 - 19(-1-1)$$

4. Simplify sing only prior axioms, theorems, and definitions:

$$5^3 + 15(-1-2)$$

5. Simplify sing only prior axioms, theorems, and definitions:

$$4^3 - 10(-3-5) + -1(13)$$

6. Simplify sing only prior axioms, theorems, and definitions:

$$1^2 - 3(4+3)$$

7. Simplify sing only prior axioms, theorems, and definitions:

$$2^4 + 8(0-11) - 2(7)$$

8. Simplify sing only prior axioms, theorems, and definitions:

$$-5^4 + 18(-5-15) + 0(5)$$

9. Simplify sing only prior axioms, theorems, and definitions:

$$5^4 - 1(2+14) - -1(6)$$

10. Simplify sing only prior axioms, theorems, and definitions:

$$-4^2 + 1(5-15)$$

2.6 Simplify

In this section, we learn no new theorems or axioms. Instead, we take some time to reflect on what we have learned so far. We will take random expressions and practice simplifying them. The goal is not so much to obtain the final expression. More than that, we want to practice our axioms, our definitions, and our theorems. Moreover, we want to practice sound reasoning and free will. You are encouraged to use your free will to do these problems as you fancy, combining the correct use of any and all the axioms and theorems as you please.

Example Compute $3-(4-5)-2^3$.
solution:

$$\begin{aligned}
&3-(4-5)-2^3 && \text{(given)}\\
&=3+-(4+-5)+-2^3 && \text{(def a-b)}\\
&=3+-1[4+(-4+-1)]+-2^3 && \text{(N+NT)}\\
&=3+-1[(4+-4)+-1]+-2^3 && \text{(ALA)}\\
&=3+-1[0+-1]+-2^3 && \text{(AInV)}\\
&=3+-1[-1]+-2^3 && \text{(Aid)}\\
&=3+1+-2^3 && \text{(NotNot)}\\
&=(3+1)+-2\cdot 2\cdot 2 && \text{(ALA, NExpo)}\\
&=4+-8 && \text{(AT, TT)}\\
&=4+(-4+-4) && \text{(N+NT)}\\
&=(4+-4)+-4 && \text{(ALA)}\\
&=0+-4 && \text{(AinV)}\\
&=-4 && \text{(Aid)}
\end{aligned}$$

Keep in mind that we would like to discourage quick methods or shortcuts for the sake of getting the answer fast. Being that we don't really care about the answer, well we don't really care if it comes fast or not. In fact, we have a slight preference for answers that do not come fast. Again, we are practicing sound reasoning, our axioms, our theorems, and our creative & free will. Moreover, we recognize this is the best time to practice these, as the future will bring us to other challenges, and this opportunity to focus intently on the axioms will be lost or diminished.

2.6.1 Exercises

1. Simplify sing only prior axioms, theorems, and definitions:

$$-4^2+11(3+15)-3(5)$$

2. Simplify sing only prior axioms, theorems, and definitions:

$$0^2+4(1-12)-2(5)$$

3. Prove that for each non-zero integer, a, there can only by one Additive Inverses, namely $-a$. (ie. show that if some other number k acts like an additive inverse, it must be equal to $-a$)

4.

a. Practice proving NoTNoT on your own. Start with $-1+1=0$ by then take careful, justified steps until you conclude $-1\cdot -1=1$

b. Practice proving NoTNoT for generic numbers, $-x$ and $-y$ on your own. Prove that $-x+-y=-(x+y)$

c. covering others' songs is great, but writing your own songs is even better. Try to write your very own proof of NoTNoT.

5. Simplify sing only prior axioms, theorems, and definitions:

$$-4^3+20(2-2)$$

6. Simplify sing only prior axioms, theorems, and definitions:

$$0^3+14(-1-14)+-2(14)$$

7. Simplify sing only prior axioms, theorems, and definitions:

$$-1^4+3(-4-9)$$

8.

a. Practice proving MT on your own. Start with $-1+1=0$ by then take careful, justified steps until you conclude $-1\cdot 5=-5$

b. Practice proving MT for generic number, x, on your own. Prove that $-1\cdot x=-x$

c. covering others' songs is great, but writing your own songs is even better. Try to write your very own proof of MT.

9. Simplify sing only prior axioms, theorems, and definitions:

$$5^2 - 11(1 + 10)$$

10. Simplify sing only prior axioms, theorems, and definitions:

$$0^3 - 13(-5 + 9)$$

11. Simplify sing only prior axioms, theorems, and definitions:

$$1^2 - 17(2 + 7) - -2(8)$$

12. Simplify sing only prior axioms, theorems, and definitions:

$$-4^2 + 20(1 + 8) - -4(3)$$

2.7 Chapter 2 Review

ESSENTIAL DEFINITIONS & THEOREMS

Definition/Theorem	Short	Example
Zero Multiplication Theorem	$[0MT]$	$3 \cdot 0 = 0$
Minus Theorem	$[MT]$	$-1 \cdot 3 = -3$
Negative times Positive Theorem	$[NPT]$	$-5 \cdot 3 = -15$
Negative One Times Negative One Theorem	$[NotNot]$	$-1 \cdot -1 = 1$
Negative times Negative Theorem	$[NNT]$	$-5 \cdot -3 = 15$
Negative Plus Negative Theorem	$[N + NT]$	$-5 + -3 = -8$
Natural Exponents	$[N.Expo]$	$5^3 = 5 \cdot 5 \cdot 5$
Just Add Exponents	$[JAE]$	$5^3 \cdot 5^4 = 5^7$
Power to Power	$[P2P]$	$(5^3)^4 = 5^{12}$
Definition of Subtraction	$[def\,a - b]$	$5 - 3 = 5 + -3$

Some Answers

Section 2.1

1. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0\\
(-1+1)48&=0\cdot 48\\
-1\cdot 48+1\cdot 48&=0\cdot 48\\
-1\cdot 48+48&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot 48+48)+-48&=0+-48 &&\text{(CLA)}\\
-1\cdot 48+(48+-48)&=0+-48 &&\text{(ALA)}\\
-1\cdot 48+0&=0+-48 &&\text{(Ainv)}\\
-1\cdot 48&=-48 &&\text{(Ainv)}
\end{aligned}$$

2. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0\\
(-1+1)8&=0\cdot 8\\
-1\cdot 8+1\cdot 8&=0\cdot 8\\
-1\cdot 8+8&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot 8+8)+-8&=0+-8 &&\text{(CLA)}\\
-1\cdot 8+(8+-8)&=0+-8 &&\text{(ALA)}\\
-1\cdot 8+0&=0+-8 &&\text{(Ainv)}\\
-1\cdot 8&=-8 &&\text{(Ainv)}
\end{aligned}$$

3. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0\\
(-1+1)50&=0\cdot 50\\
-1\cdot 50+1\cdot 50&=0\cdot 50\\
-1\cdot 50+50&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot 50+50)+-50&=0+-50 &&\text{(CLA)}\\
-1\cdot 50+(50+-50)&=0+-50 &&\text{(ALA)}\\
-1\cdot 50+0&=0+-50 &&\text{(Ainv)}\\
-1\cdot 50&=-50 &&\text{(Ainv)}
\end{aligned}$$

Section 2.2

1. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0 &&\text{(ainv)}\\
(-1+1)33&=0\cdot 33 &&\text{(clm)}\\
-1\cdot 33+1\cdot 33&=0\cdot 33 &&\text{(dl)}\\
-1\cdot 33+33&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot 33+33)+-33&=0+-33 &&\text{(CLA)}\\
-1\cdot 33+(33+-33)&=0+-33 &&\text{(ALA)}\\
-1\cdot 33+0&=0+-33 &&\text{(Ainv)}\\
-1\cdot 33&=-33 &&\text{(Ainv)}
\end{aligned}$$

2. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0 &&\text{(ainv)}\\
(-1+1)31&=0\cdot 31 &&\text{(clm)}\\
-1\cdot 31+1\cdot 31&=0\cdot 31 &&\text{(dl)}\\
-1\cdot 31+31&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot 31+31)+-31&=0+-31 &&\text{(CLA)}\\
-1\cdot 31+(31+-31)&=0+-31 &&\text{(ALA)}\\
-1\cdot 31+0&=0+-31 &&\text{(Ainv)}\\
-1\cdot 31&=-31 &&\text{(Ainv)}
\end{aligned}$$

3. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0 &&\text{(ainv)}\\
(-1+1)22&=0\cdot 22 &&\text{(clm)}\\
-1\cdot 22+1\cdot 22&=0\cdot 22 &&\text{(dl)}\\
-1\cdot 22+22&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot 22+22)+-22&=0+-22 &&\text{(CLA)}\\
-1\cdot 22+(22+-22)&=0+-22 &&\text{(ALA)}\\
-1\cdot 22+0&=0+-22 &&\text{(Ainv)}\\
-1\cdot 22&=-22 &&\text{(Ainv)}
\end{aligned}$$

4. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0 &&\text{(ainv)}\\
(-1+1)u&=0\cdot u &&\text{(clm)}\\
-1\cdot u+1\cdot u&=0\cdot u &&\text{(dl)}\\
-1\cdot u+u&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot u+u)+-u&=0+-u &&\text{(CLA)}\\
-1\cdot u+(u+-u)&=0+-u &&\text{(ALA)}\\
-1\cdot u+0&=0+-u &&\text{(Ainv)}\\
-1\cdot u&=-u &&\text{(Ainv)}
\end{aligned}$$

5. one way to prove it is to begin with $-1+1=0$

$$\begin{aligned}
-1+1&=0 &&\text{(ainv)}\\
(-1+1)w&=0\cdot w &&\text{(clm)}\\
-1\cdot w+1\cdot w&=0\cdot w &&\text{(dl)}\\
-1\cdot w+w&=0 &&\text{(Mid, 0MT)}\\
(-1\cdot w+w)+-w&=0+-w &&\text{(CLA)}\\
-1\cdot w+(w+-w)&=0+-w &&\text{(ALA)}\\
-1\cdot w+0&=0+-w &&\text{(Ainv)}\\
-1\cdot w&=-w &&\text{(Ainv)}
\end{aligned}$$

6.

$$\begin{aligned}
&-6\cdot 2 &&\text{(given)}\\
&=(-1\cdot 6)\cdot 2 &&\text{(MT)}\\
&=-1(6\cdot 2) &&\text{(ALM)}\\
&=-1\cdot 12 &&\text{(ALM)}\\
&=-12 &&\text{(MT)}
\end{aligned}$$

7.

$$\begin{aligned}
&-13\cdot 4 &&\text{(given)}\\
&=(-1\cdot 13)\cdot 4 &&\text{(MT)}\\
&=-1(13\cdot 4) &&\text{(ALM)}\\
&=-1\cdot 52 &&\text{(ALM)}\\
&=-52 &&\text{(MT)}
\end{aligned}$$

8.

$$\begin{aligned}
&-8\cdot 7 &&\text{(given)}\\
&=(-1\cdot 8)\cdot 7 &&\text{(MT)}\\
&=-1(8\cdot 7) &&\text{(ALM)}\\
&=-1\cdot 56 &&\text{(ALM)}\\
&=-56 &&\text{(MT)}
\end{aligned}$$

9.

$$\begin{aligned}
&-13\cdot -5 &&\text{(given)}\\
&=(-1\cdot 13)\cdot(-1\cdot 5) &&\text{(MT)}\\
&=(-1\cdot -1)(13\cdot 5) &&\text{(ALM, CoLM)}\\
&=1\cdot 65 &&\text{(NoTNoT, TT)}\\
&=65 &&\text{(MT)}
\end{aligned}$$

10.

$$\begin{aligned}
&-6\cdot -2 &&\text{(given)}\\
&=(-1\cdot 6)\cdot(-1\cdot 2) &&\text{(MT)}\\
&=(-1\cdot -1)(6\cdot 2) &&\text{(ALM, CoLM)}\\
&=1\cdot 12 &&\text{(NoTNoT, TT)}\\
&=12 &&\text{(MT)}
\end{aligned}$$

11.

$$\begin{aligned}
&-13\cdot -7 &&\text{(given)}\\
&=(-1\cdot 13)\cdot(-1\cdot 7) &&\text{(MT)}\\
&=(-1\cdot -1)(13\cdot 7) &&\text{(ALM, CoLM)}\\
&=1\cdot 91 &&\text{(NoTNoT, TT)}\\
&=91 &&\text{(MT)}
\end{aligned}$$

Section 2.3

1.

$$\begin{aligned} &-14+8 \\ &=(-6+-8)+8 && \text{(N + NT)} \\ &=-6+(-8+8) && \text{(ALA)} \\ &=-6+(0) && \text{(AInV)} \\ &=-6 && \text{(AiD)} \end{aligned}$$

2.

$$\begin{aligned} &-10+3 \\ &=(-7+-3)+3 && \text{(N + NT)} \\ &=-7+(-3+3) && \text{(ALA)} \\ &=-7+(0) && \text{(AInV)} \\ &=-7 && \text{(AiD)} \end{aligned}$$

3.

$$\begin{aligned} &-23+11 \\ &=(-12+-11)+11 && \text{(N + NT)} \\ &=-12+(-11+11) && \text{(ALA)} \\ &=-12+(0) && \text{(AInV)} \\ &=-12 && \text{(AiD)} \end{aligned}$$

4.

$$\begin{aligned} &2+-3 \\ &=2+(-2+-1) && \text{(N + NT)} \\ &=(-2+2)+-1 && \text{(ALA)} \\ &=0+-1 && \text{(AInV)} \\ &=-1 && \text{(AiD)} \end{aligned}$$

5.

$$\begin{aligned} &13+-16 \\ &=13+(-13+-3) && \text{(N + NT)} \\ &=(-13+13)+-3 && \text{(ALA)} \\ &=0+-3 && \text{(AInV)} \\ &=-3 && \text{(AiD)} \end{aligned}$$

6.

$$\begin{aligned} &12+-23 \\ &=12+(-12+-11) && \text{(N + NT)} \\ &=(-12+12)+-11 && \text{(ALA)} \\ &=0+-11 && \text{(AInV)} \\ &=-11 && \text{(AiD)} \end{aligned}$$

7.

$$\begin{aligned} &-14+-12 && \text{(given)} \\ &=-1\cdot 14+-1\cdot 12 && \text{(MT)} \\ &=-1(14+12) && \text{(DL)} \\ &=-1\cdot 26 && \text{(AT)} \\ &=-26 && \text{(MT)} \end{aligned}$$

8.

$$\begin{aligned} &-3+-5 && \text{(given)} \\ &=-1\cdot 3+-1\cdot 5 && \text{(MT)} \\ &=-1(3+5) && \text{(DL)} \\ &=-1\cdot 8 && \text{(AT)} \\ &=-8 && \text{(MT)} \end{aligned}$$

9.

$$\begin{aligned} &-7+-13 && \text{(given)} \\ &=-1\cdot 7+-1\cdot 13 && \text{(MT)} \\ &=-1(7+13) && \text{(DL)} \\ &=-1\cdot 20 && \text{(AT)} \\ &=-20 && \text{(MT)} \end{aligned}$$

10.

$$\begin{aligned} &-c+-x && \text{(given)} \\ &=-1\cdot c+-1\cdot x && \text{(MT)} \\ &=-1(c+x) && \text{(DL)} \end{aligned}$$

11.

$$\begin{aligned} &-c+-x && \text{(given)} \\ &=-1\cdot c+-1\cdot x && \text{(MT)} \\ &=-1(c+x) && \text{(DL)} \end{aligned}$$

12. answers may have many variation, make sure to take beatiful steps and reasoning to arrive at 38

13. answers may have many variation, make sure to take beatiful steps and reasoning to arrive at -16

14. answers may have many variation, make sure to take beatiful steps and reasoning to arrive at -65

15. answers may have many variation, make sure to take beatiful steps and reasoning to arrive at 5

16. answers may have many variation, make sure to take beatiful steps and reasoning to arrive at -1

Section 2.4

1. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -37

2. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -12

3. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -20

4. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -21

5. many ways to do it, here is one approach:

$$\begin{aligned} &19-13(5-6)-8 && \text{(given)} \\ &=19+-13(5+-6)+-8 && \text{(def a-b)} \\ &=19+-13(5)+-13(-6)+-8 && \text{(DL)} \\ &=19+-65+78+-8 && \text{(NPT, NNT)} \\ &=(19+78)+(-65+-8) && \text{(ALA, CoLA)} \\ &=97+-73 && \text{(At, N+NT)} \\ &=(24+73)+-73 && \text{(AT)} \\ &=24+(73+-73) && \text{(ALA)} \\ &=24 && \text{(AinV, Aid)} \end{aligned}$$

6. many ways to do it, here is one approach:

$$\begin{aligned} &20-3(3-4)-21 && \text{(given)} \\ &=20+-3(3+-4)+-21 && \text{(def a-b)} \\ &=20+-3(3)+-3(-4)+-21 && \text{(DL)} \\ &=20+-9+12+-21 && \text{(NPT, NNT)} \\ &=(20+12)+(-9+-21) && \text{(ALA, CoLA)} \\ &=32+-30 && \text{(At, N+NT)} \\ &=(2+30)+-30 && \text{(AT)} \\ &=2+(30+-30) && \text{(ALA)} \\ &=2 && \text{(AinV, Aid)} \end{aligned}$$

7. many ways to do it, here is one approach:

$$\begin{aligned} &14-10(1-2)-14 && \text{(given)} \\ &=14+-10(1+-2)+-14 && \text{(def a-b)} \\ &=14+-10(1)+-10(-2)+-14 && \text{(DL)} \\ &=14+-10+20+-14 && \text{(NPT, NNT)} \\ &=(14+20)+(-10+-14) && \text{(ALA, CoLA)} \\ &=34+-24 && \text{(At, N+NT)} \\ &=(10+24)+-24 && \text{(AT)} \\ &=10+(24+-24) && \text{(ALA)} \\ &=10 && \text{(AinV, Aid)} \end{aligned}$$

8. many ways to do it, here is one approach:

$$\begin{aligned} &16-1(29-6)-23 && \text{(given)} \\ &=16+-1(29+-6)+-23 && \text{(def a-b)} \\ &=16+-1(29)+-1(-6)+-23 && \text{(DL)} \\ &=16+-29+6+-23 && \text{(NPT, NNT)} \\ &=(16+6)+(-29+-23) && \text{(ALA, CoLA)} \\ &=22+-52 && \text{(At, N+NT)} \\ &=22+(-22+-30) && \text{(N+NT)} \\ &=(22+-22)+-30 && \text{(ALA)} \\ &=-30 && \text{(AinV, Aid)} \end{aligned}$$

9. many ways to do it, here is one approach:

$$
\begin{aligned}
&25 - 9(24 - 2) - 22 && \text{(given)}\\
=&25 + -9(24 + -2) + -22 && \text{(def a-b)}\\
=&25 + -9(24) + -9(-2) + -22 && \text{(DL)}\\
=&25 + -216 + 18 + -22 && \text{(NPT, NNT)}\\
=&(25 + 18) + (-216 + -22) && \text{(ALA, CoLA)}\\
=&43 + -238 && \text{(At, N+NT)}\\
=&43 + (-43 + -195) && \text{(N+NT)}\\
=&(43 + -43) + -195 && \text{(ALA)}\\
=&-195 && \text{(AinV, Aid)}
\end{aligned}
$$

10. many ways to do it, here is one approach:

$$
\begin{aligned}
&27 - 12(23 - 3) - 6 && \text{(given)}\\
=&27 + -12(23 + -3) + -6 && \text{(def a-b)}\\
=&27 + -12(23) + -12(-3) + -6 && \text{(DL)}\\
=&27 + -276 + 36 + -6 && \text{(NPT, NNT)}\\
=&(27 + 36) + (-276 + -6) && \text{(ALA, CoLA)}\\
=&63 + -282 && \text{(At, N+NT)}\\
=&63 + (-63 + -219) && \text{(N+NT)}\\
=&(63 + -63) + -219 && \text{(ALA)}\\
=&-219 && \text{(AinV, Aid)}
\end{aligned}
$$

11. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -29

12. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -153

13. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -85

14. answers may have many variation, make sure to take tight steps and flawless reasoning to arrive at -16

Section 2.5

1. answers may vary, stop and smell the axioms.. take your time to arrive at 126

2. answers may vary, stop and smell the axioms.. take your time to arrive at 104

3. answers may vary, stop and smell the axioms.. take your time to arrive at 294

4. answers may vary, stop and smell the axioms.. take your time to arrive at 80

5. answers may vary, stop and smell the axioms.. take your time to arrive at 131

6. answers may vary, stop and smell the axioms.. take your time to arrive at -20

7. answers may vary, stop and smell the axioms.. take your time to arrive at -86

8. answers may vary, stop and smell the axioms.. take your time to arrive at -985

9. answers may vary, stop and smell the axioms.. take your time to arrive at 615

10. answers may vary, stop and smell the axioms.. take your time to arrive at -26

Section 2.6

1. answers may vary, stop and smell the axioms.. take your time to arrive at 167

2. answers may vary, stop and smell the axioms.. take your time to arrive at -54

3. this is a very nice problem, you should think about it..

4. think about it

5. answers may vary, stop and smell the axioms.. take your time to arrive at -64

6. answers may vary, stop and smell the axioms.. take your time to arrive at -238

7. answers may vary, stop and smell the axioms.. take your time to arrive at -40

8. think about it

9. answers may vary, stop and smell the axioms.. take your time to arrive at -96

10. answers may vary, stop and smell the axioms.. take your time to arrive at -52

11. answers may vary, stop and smell the axioms.. take your time to arrive at -136

12. answers may vary, stop and smell the axioms.. take your time to arrive at 176

Bibliography

[1] Wikipedia Euclid's Elements Retrieved August 10, 2011

[2] Wikipedia Georg Cantor Retrieved August 2011

[3] Wikipedia Cantor's Diagonal Argument Retrieved August 2011

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