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CHAPTER 3

# INTRODUCTION TO RATIONALS

Our first chapter covered the basic axioms that we intend to use for the throughout the entire course.

In our second chapter we learned all the basic properties and important theorems explaining how we multiply and/or add integer numbers. Said differently, we became acquainted with the basic binary operations, addition and multiplication over the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

This will be a reoccurring theme throughout the course. We will often meet a new family of numbers then carefully and methodically learn to add and multiply such numbers. Ultimately we will turn our adding and multiplying skills into powerful equation-solving skills, which is the quintessential algebraic skills everyone seeks.

That being said, we now turn our attention to a new family of numbers. This time we learn about *fraction*s. By fractions we mean numbers that can be written with a couple *integers* stacked one over the other, such as

$$\frac{3}{5}$$

The only restriction is that the top number, call the numerator, be an integer, and the bottom number, called the denominator, be a non-zero integer. A common name for the collection of all such possible fractions is the set of *Rational Number*s. The symbol $\mathbb{Q}$ is often use to denote the set of rational numbers. We could abbreviate the definition of the set of rational numbers, as the set of all possible numbers $\frac{a}{b}$ such that $a \in \mathbb{Z}$ [read "$a$ is an *element of* the set of integers" or "$a$ is an integer"] and $b \in \mathbb{Z}\backslash\{0\}$ [read, $b$ is an integer, but not zero].

Famous examples of rational numbers include

$$\frac{1}{2}, \frac{3}{5}, \frac{7}{3}, etc.$$

A famous example of a number that can not be written as an integer fraction is $\pi$, and we will soon meet other irrationals. Be that as it may, our work for now has been cut out. We intend to delicately learn all the ways of the rationals, how to add them, multiply them, simplify them and more. In doing so, we will learn much about *polynomials* and other creatures.

## 3.1 Fractions

We have already met some fractions. Namely the ones with a '1' on th enumerator, such as $\frac{1}{3}$ and $\frac{1}{5}$. We would now like to define what we mean when we write more general fractions, such as ones that have something else other than 1 on the top. We do precisely this using a couple definitions.

**Definition** (def a/b) If $b \neq 0$ we define the *fraction* $\frac{a}{b}$ as

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

Consider for example, the fraction

$$\frac{5}{3}$$

. According to our proposed definition, $\frac{5}{3}$ is defined to be

$$\frac{5}{3} = 5 \cdot \frac{1}{3}$$

This means it is equal to 5 one thirds, or 5 multiplicative inverses of 3. This turns out to be a most powerful way view of fractions. Let us practice a couple times.

**Examples**

$$\frac{4}{3} = 4 \cdot \frac{1}{3} \qquad \text{(def a/b)}$$

$$\frac{5}{3} = 5 \cdot \frac{1}{7} \qquad \text{(def a/b)}$$

$$\frac{-7}{x} = -7 \cdot \frac{1}{x} \qquad \text{(def a/b)}$$

$$\frac{-7-y}{x^2+1} = (-7-y) \cdot \frac{1}{x^2+1} \qquad \text{(def a/b)}$$

$$3 \cdot \frac{1}{13} = \frac{3}{13} \qquad \text{(def a/b)}$$

Next, let us consider a special example, consider what is $\frac{3}{3}$? according to this definition,

$$\frac{3}{3} = 3 \cdot \frac{1}{3}$$

but we already knew what $\frac{1}{3}$ represents, it is the multiplicative inverse of 3. Thus,

$$\begin{aligned} &\frac{3}{3} && \text{(given)} \\ &= 3 \cdot \frac{1}{3} && \text{(def a/b)} \\ &= 1 && \text{(MinV)} \end{aligned}$$

This example is so important we will make a theorem out of it.

**Theorem 3.1.1.** *Just One Theorem [JOT] If $a$ is a non-zero number for which our axioms apply, then*

$$\frac{a}{a} = 1$$

Moreover, we offer yet another way to look at it.

$$\begin{aligned} 1 &= \frac{3}{3} && \text{(JOT)} \\ &= 3 \cdot \frac{1}{3} && \text{(def a/b)} \\ &= (1+1+1) \cdot \frac{1}{3} && \text{(AT)} \\ &= 1 + \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} && \text{(DL)} \\ &= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} && \text{(MiD)} \end{aligned}$$

This implies by [TP] that

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

which means that the number, $\frac{1}{3}$, added to itself 3 times yields '1' whole number. Thus another characterization of the number $\frac{1}{3}$ and another way to see the implications of:

$$\frac{3}{3} = 3 \cdot \frac{1}{3} = 1$$

Now, suppose we collected all possible fractions of the type $a/b$ as we have just described and place them into one bucket, one set. Such a set is called the set of rational numbers. Formally we have,

**Definition** A number $x$ will be called a *Rational Number* if it can be written as

$$x = \frac{a}{b}$$

where $a, b \in \mathbb{Z}$ and $b \neq 0$

It is important to note that the prior family of number we meant, $\mathbb{Z}$ is a subset of this larger family of numbers $\mathbb{Q}$. Said differently, every integer is also a rational number. Yet another way to say it is, "every integer can be written as an integer fraction". For example consider the integer 7. Then..

$$\begin{aligned} 7 &= 7 \cdot 1 && \text{(MiD)} \\ &= 7 \cdot \frac{1}{1} && \text{(JOT)} \\ &= \frac{7}{1} && \text{(def a/b)} \end{aligned}$$

Thus $7 = \frac{7}{1}$ a fraction, therefore 7 is a rational number, ie

$$7 \in \mathbb{Q}$$

. That leaves us with two clear statements. One, that any integer can be written as a fraction, thus every integer is also a rational number. Second, every number can be written with a "one under". We shall make this last statement into a theorem.

**Theorem 3.1.2.** *One Under Theorem [OUT] If $a$ is a number for which our axioms apply, then*

$$a = \frac{a}{1}$$

### 3.1.1 Exercises

**1.**

Select the statement/s that can be justified by [OUT]

A) $\frac{2}{2} = 1$ B) $\frac{\frac{1}{20}}{1} = \frac{1}{20}$ C) $8 = \frac{8}{1}$ D) $2 = 2 \cdot 1$ E) $2 = \frac{2}{1}$ F) $\frac{-5}{1} = -5$
G) $\alpha = \frac{\alpha}{1}$ H) $\frac{0}{0} = 1$ I) $\frac{3}{3} = 1$ J) none of these

---

**2.**

Select the statement/s that can be justified by [Def a-b]

A) $\frac{w}{w} = 1$ B) $\frac{6}{-2} = 6 \cdot \frac{1}{-2}$ C) $\frac{-1}{\frac{1}{8}} = -1 \cdot \frac{1}{\frac{1}{8}}$ D) $\frac{\frac{1}{8}}{\alpha} = \frac{1}{8} \cdot \frac{1}{\alpha}$ E) none of these

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**3.**

Select the statement/s that can be justified by [JOT]

A) $\frac{0}{0} = 1$ B) $\frac{-4}{\frac{1}{15}} = -4 \cdot \frac{1}{\frac{1}{15}}$ C) $\frac{y}{y} = 1$ D) $\frac{3}{3} = 1$ E) none of these

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**4.**

Select the statement/s that can be justified by [JOT]

A) $\frac{v}{v} = 1$ B) $\frac{8}{8} = 1$ C) $\frac{-5}{\frac{1}{14}} = -5 \cdot \frac{1}{\frac{1}{14}}$ D) $\frac{0}{0} = 1$ E) none of these

---

**5.**

Select the statement/s that can be justified by [OUT]

A) $\frac{2}{2} = 1$ B) $\frac{\frac{13}{58}}{1} = \frac{13}{58}$ C) $2 = \frac{2}{1}$ D) $\frac{0}{0} = 1$ E) $\alpha = \frac{\alpha}{1}$ F) $2 = \frac{2}{1}$ G) $2 = 2 \cdot 1$
H) $\frac{2}{2} = 1$ I) $\frac{-4}{1} = -4$ J) none of these

---

**6.**

Select the statement/s that can be justified by [Def a-b]

A) $\frac{6}{-13} = 6 \cdot \frac{1}{-13}$ B) $\frac{3}{\frac{17}{44}} = 3 \cdot \frac{1}{\frac{17}{44}}$ C) $\frac{w}{w} = 1$ D) $\frac{\frac{17}{44}}{\text{☯}} = \frac{17}{44} \cdot \frac{1}{\text{☯}}$ E) none of these

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**7.**

Select the statement/s that can be justified by [Def a-b]

A) $\frac{-2}{-16} = -2 \cdot \frac{1}{-16}$ B) $\frac{5}{\frac{8}{11}} = 5 \cdot \frac{1}{\frac{8}{11}}$ C) $\frac{t}{t} = 1$ D) $\frac{\frac{8}{11}}{\text{☕}} = \frac{8}{11} \cdot \frac{1}{\text{☕}}$ E) none of these

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**8.**

Select the statement/s that can be justified by [OUT]

A) $\frac{\frac{9}{20}}{1} = \frac{9}{20}$ B) $2 = \frac{2}{1}$ C) $\frac{3}{3} = 1$ D) $3 = \frac{3}{1}$ E) $\frac{7}{7} = 1$ F) $\frac{7}{1} = 7$ G) $\frac{0}{0} = 1$
H) $3 = 3 \cdot 1$ I) $\text{🚲} = \frac{\text{🚲}}{1}$ J) none of these

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**9.**

Select the statement/s that can be justified by [JOT]

A) $\frac{7}{7} = 1$ B) $\frac{6}{\frac{1}{2}} = 6 \cdot \frac{1}{\frac{1}{2}}$ C) $\frac{w}{w} = 1$ D) $\frac{0}{0} = 1$ E) none of these

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**10.**

Select the statement/s that can be justified by [OUT]

A) $9 = 9 \cdot 1$ B) $9 = \frac{9}{1}$ C) $\frac{9}{9} = 1$ D) $\text{🐴} = \frac{\text{🐴}}{1}$ E) $4 = \frac{4}{1}$ F) $\frac{\frac{17}{43}}{1} = \frac{17}{43}$ G) $\frac{6}{1} = 6$
H) $\frac{3}{3} = 1$ I) $\frac{0}{0} = 1$ J) none of these

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**11.**

Select the statement/s that can be justified by [JOT]

A) $\frac{\frac{8}{13}}{\alpha} = \frac{8}{13} \cdot \frac{1}{\alpha}$ B) $\frac{5}{5} = 1$ C) $\frac{4}{\frac{8}{13}} = 4 \cdot \frac{1}{\frac{8}{13}}$ D) $\frac{e}{e} = 1$ E) none of these

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**12.**

Select the statement/s that can be justified by [OUT]

A) $\text{🐴} = \frac{\text{🐴}}{1}$ B) $\frac{0}{0} = 1$ C) $\frac{5}{5} = 1$ D) $\frac{3}{3} = 1$ E) $3 = 3 \cdot 1$ F) $\frac{\frac{17}{26}}{1} = \frac{17}{26}$
G) $\frac{-5}{1} = -5$ H) $3 = \frac{3}{1}$ I) $5 = \frac{5}{1}$ J) none of these

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**13.**

Select the statement/s that can be justified by [Def a-b]

A) $\frac{\sqrt{5}}{\sqrt{5}} = 1$ B) $\frac{0}{-3} = 0 \cdot \frac{1}{-3}$ C) $\frac{\frac{5}{16}}{\zeta} = \frac{5}{16} \cdot \frac{1}{\zeta}$ D) $\frac{2}{\frac{5}{16}} = 2 \cdot \frac{1}{\frac{5}{16}}$ E) none of these

---

**14.**

Select the statement/s that can be justified by [Def a-b]

A) $\frac{-4}{-10} = -4 \cdot \frac{1}{-10}$ B) $\frac{-7}{\frac{3}{14}} = -7 \cdot \frac{1}{\frac{3}{14}}$ C) $\frac{\pi}{\pi} = 1$ D) $\frac{\frac{3}{14}}{\heartsuit} = \frac{3}{14} \cdot \frac{1}{\heartsuit}$ E) none of these

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**15.**

Select the statement/s that can be justified by [JOT]

A) $\frac{\frac{1}{5}}{\text{☕}} = \frac{1}{5} \cdot \frac{1}{\text{☕}}$ B) $\frac{4}{4} = 1$ C) $\frac{u}{u} = 1$ D) $\frac{-5}{\frac{1}{5}} = -5 \cdot \frac{1}{\frac{1}{5}}$ E) none of these

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**16.** Prove [JOT], that is prove, for a non-zero $a$,

$$\frac{a}{a} = 1$$

**17.** Prove [OUT], that is prove ,

$$\frac{a}{1} = a$$

## 3.2 Multiplying Rational Numbers

Last section we introduce the rational numbers along with some very mild and modest theorems, JOT & OUT. In this section, we dive into the very heart of fraction multiplying. Essentially, this is done with two theorems. The first theorem concerns itself with multiplying fractions only of the type $\frac{1}{a}$. In other words, our first theorem explains only how to multiply only fractions that have a 1 in the numerator. Formally, we state:

**Theorem 3.2.1.** *Multiply Bottoms Theorem [MBT] If $a$ and $b$ and $ab$ are non-zero number for which our axioms apply, then*

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$$

*when $a$ and $b$ are natural numbers, we may include the TT step with this theorem.*

*Proof.*

$$\begin{aligned}
1 &= ab \cdot \frac{1}{ab} && \text{(Minv)}\\
\left(\frac{1}{a} \cdot \frac{1}{b}\right) \cdot 1 &= \left(\frac{1}{a} \cdot \frac{1}{b}\right) \cdot ab \cdot \frac{1}{ab} && \text{(CLM)}\\
\frac{1}{a} \cdot \frac{1}{b} &= \left(\frac{1}{a} \cdot \frac{1}{b}\right) \cdot ab \cdot \frac{1}{ab} && \text{(MiD)}\\
\frac{1}{b} \cdot \frac{1}{a} &= \left(\frac{1}{b} \cdot b\right) \left(\frac{1}{a} \cdot a\right) \frac{1}{ab} && \text{(ColM, AlM)}\\
\frac{1}{a} \cdot \frac{1}{b} &= (1)\,(1)\,\frac{1}{ab} && \text{(MinV)}\\
\frac{1}{a} \cdot \frac{1}{b} &= \frac{1}{ab} && \text{(MiD)}
\end{aligned}$$

∎

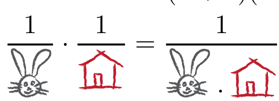
**Examples** Using the MBT..

$$\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15} \qquad \text{(MBT [w TT included])}$$

$$\frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x} \qquad \text{(MBT)}$$

$$\frac{1}{-3} \cdot \frac{1}{5x^3} = \frac{1}{-3 \cdot 5x^3} \qquad \text{(MBT)}$$

$$\frac{1}{x+2} \cdot \frac{1}{5x^3 - x} = \frac{1}{(x+2)(5x^3 - x)} \qquad \text{(MBT)}$$



(MBT)

That's how we multiply fractions with numerator '1'. The next logical question might be how then, do we multiply fractions with numerators other than one. That question is addressed precisely by our next theorem.

**Theorem 3.2.2.** *Multiply Across Theorem [MAT] If $a, b, c$ and $d$ are numbers for which our axioms apply and $b, d$ are non-zero, then*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

The proof is left as a very important and doable exercise for the reader to embrace.

**Examples** Using the MAT.

$$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15} \qquad \text{(MAT [w TT included])}$$

$$\frac{5}{3} \cdot \frac{y}{x} = \frac{5y}{3x} \qquad \text{(MAT)}$$

$$\frac{-4}{-3} \cdot \frac{2}{5x^3} = \frac{-4 \cdot 2}{-3 \cdot 5x^3} \qquad \text{(MAT)}$$

$$\frac{z}{x+2} \cdot \frac{w^3}{5x^3 - x} = \frac{z \cdot w^3}{(x+2)(5x^3 - x)} \qquad \text{(MAT)}$$

$$\frac{A \cdot B}{K \cdot M} = \frac{A}{K} \cdot \frac{B}{M} \qquad \text{(MAT [backwards])}$$

At last, in this section, we take the opportunity to consider a common type of question about signs. For example, how does $\frac{-2}{3}$ compare to $\frac{2}{-3}$ or to $-\frac{2}{3}$. Out next theorem proposes that these are all equal. On a superficial level we can say that

$$\frac{-2}{3} = \frac{2}{-3}$$

means we can take the negative from the top and put it on the bottom number, or if we read it from right to left, it says a negative from the bottom can be placed on the numerator. A slightly more amusing way to see it using the definition of fractions.

$$\frac{-2}{3} = \frac{2}{-3}$$

becomes

$$-2 \cdot \frac{1}{3} = 2 \cdot \frac{1}{-3}$$

which becomes "the additive inverse of 2 times the mult. inverse of 3 *is the same as* 2 times the multiplicative inverse of the additive inverse of 3 "

Formally, we state:

**Theorem 3.2.3.** *Negative Wherever you Want Theorem [NWW] If $a$ and $b$ are numbers for which our axioms apply and $b$ is non-zero, then*

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

We leave the proof of this as an excellent exercise for the student to complete.

### 3.2.1 Exercises

**1.**

Select the statement/s that can be justified by [MBT]

A) $\frac{5}{8} \cdot \frac{7}{7} = \frac{5 \cdot 7}{8 \cdot 7}$ B) $\frac{1}{5} \cdot \frac{1}{-14} = \frac{1+1}{5+-14}$ C) $\frac{1 \cdot 1}{f \cdot v} = \frac{1}{f} \cdot \frac{1}{v}$ D) $\frac{6}{4} \cdot \frac{7}{3} = \frac{6 \cdot 7}{4 \cdot 3}$
E) $\frac{1}{5} \cdot \frac{1}{-14} = \frac{1}{5 \cdot -14}$ F) none of these

---

**2.**

Select all expression/s equivalent to:

$$\frac{1}{7} \cdot \frac{1}{20}$$

A) $1 \cdot 1 \cdot \frac{1}{7 \cdot 20}$ B) $\frac{1}{7 \cdot 20}$ C) $\frac{1}{7} \cdot \frac{1}{20} \cdot 1$ D) $\frac{1}{7} \cdot \frac{1}{20} \cdot \left(7 \cdot 20 \cdot \frac{1}{7 \cdot 20}\right)$
E) $\left(7 \cdot \frac{1}{7}\right) \cdot \left(20 \cdot \frac{1}{20}\right) \cdot \frac{1}{7 \cdot 20}$ F) these choices outline a proof of $\frac{1}{7} \cdot \frac{1}{20} = \frac{1}{7 \cdot 20}$
G) none of these

---

**3.**

Select all expression/s equivalent to:

$$\frac{-9}{9}$$

A) these choices outline a proof of $\frac{-9}{9} = -\frac{9}{9}$ B) $-1 \cdot \left(\frac{9}{9}\right)$ C) $-\frac{9}{9}$
D) $-1 \cdot 9 \cdot \frac{1}{9}$ E) $-9 \cdot \frac{1}{9}$ F) $-1 \cdot \left(9 \cdot \frac{1}{9}\right)$ G) none of these

---

**4.**

Select the statement/s that can be justified by [MBT]
A) $\frac{1}{6} \cdot \frac{1}{-6} = \frac{1+1}{6+-6}$ B) $\frac{10}{4} \cdot \frac{3}{9} = \frac{10 \cdot 3}{4 \cdot 9}$ C) $\frac{1}{6} \cdot \frac{1}{-6} = \frac{1}{6 \cdot -6}$ D) $\frac{6}{5} \cdot \frac{9}{9} = \frac{6 \cdot 9}{5 \cdot 9}$
E) $\frac{1 \cdot 1}{h \cdot v} = \frac{1}{h} \cdot \frac{1}{v}$ F) none of these

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**5.**

Select all expression/s equivalent to:

$$\frac{-8}{16}$$

A) $-1 \cdot \left(\frac{8}{16}\right)$ B) these choices outline a proof of $\frac{-8}{16} = -\frac{8}{16}$ C) $-\frac{8}{16}$
D) $-1 \cdot \left(8 \cdot \frac{1}{16}\right)$ E) $-1 \cdot 8 \cdot \frac{1}{16}$ F) $-8 \cdot \frac{1}{16}$ G) none of these

---

**6.**

Select the statement/s that can be justified by [MBT]
A) $\frac{1}{r \cdot x} = \frac{1}{r} \cdot \frac{1}{x}$ B) $\frac{1}{10} \cdot \frac{1}{2} = \frac{1}{10 \cdot 2}$ C) $\frac{1}{4 \cdot 7} = \frac{1}{4} \cdot \frac{1}{7}$ D) $\frac{1}{2 \cdot 4} = \frac{1}{2} \cdot \frac{1}{4}$
E) $\frac{1}{4} \cdot \frac{1}{-1} = \frac{1+1}{4+-1}$ F) $\frac{1}{4} \cdot \frac{1}{-1} = \frac{1}{4 \cdot -1}$ G) none of these

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**7.**

Select all expression/s equivalent to:

$$\frac{-10}{19}$$

A) $-1 \cdot \left(\frac{10}{19}\right)$ B) $-1 \cdot 10 \cdot \frac{1}{19}$ C) these choices outline a proof of $\frac{-10}{19} = -\frac{10}{19}$
D) $-\frac{10}{19}$ E) $-10 \cdot \frac{1}{19}$ F) $-1 \cdot \left(10 \cdot \frac{1}{19}\right)$ G) none of these

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**8.**

Select all expression/s equivalent to:

$$\frac{1}{f} \cdot \frac{1}{t}$$

A) these choices outline a proof of $\frac{1}{f} \cdot \frac{1}{t} = \frac{1}{f \cdot t}$ B) $\left(f \cdot \frac{1}{f}\right) \cdot \left(t \cdot \frac{1}{t}\right) \cdot \frac{1}{f \cdot t}$
C) $\frac{1}{f \cdot t}$ D) $\frac{1}{f} \cdot \frac{1}{t} \cdot 1$ E) $1 \cdot 1 \cdot \frac{1}{f \cdot t}$ F) $\frac{1}{f} \cdot \frac{1}{t} \cdot \left(f \cdot t \cdot \frac{1}{f \cdot t}\right)$ G) none of these

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**9.**

Select all expression/s equivalent to:

$$\frac{1}{7} \cdot \frac{1}{7}$$

A) $\left(7 \cdot \frac{1}{7}\right) \cdot \left(7 \cdot \frac{1}{7}\right) \cdot \frac{1}{7 \cdot 7}$ B) these choices outline a proof of $\frac{1}{7} \cdot \frac{1}{7} = \frac{1}{7 \cdot 7}$
C) $\frac{1}{7} \cdot \frac{1}{7} \cdot 1$ D) $\frac{1}{7} \cdot \frac{1}{7} \cdot \left(7 \cdot 7 \cdot \frac{1}{7 \cdot 7}\right)$ E) $1 \cdot 1 \cdot \frac{1}{7 \cdot 7}$ F) $\frac{1}{7 \cdot 7}$ G) none of these

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**10.**

Select all expression/s equivalent to:

$$\frac{1}{g} \cdot \frac{1}{x}$$

A) $\frac{1}{g} \cdot \frac{1}{x} \cdot \left(g \cdot x \cdot \frac{1}{g \cdot x}\right)$ B) $1 \cdot 1 \cdot \frac{1}{g \cdot x}$ C) $\left(g \cdot \frac{1}{g}\right) \cdot \left(x \cdot \frac{1}{x}\right) \cdot \frac{1}{g \cdot x}$ D) $\frac{1}{g \cdot x}$
E) $\frac{1}{g} \cdot \frac{1}{x} \cdot 1$ F) these choices outline a proof of $\frac{1}{g} \cdot \frac{1}{x} = \frac{1}{g \cdot x}$ G) none of these

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**11.**

Select all expression/s equivalent to:

$$\frac{1}{y} \cdot \frac{1}{x}$$

A) $\frac{1}{y \cdot x}$ B) these choices outline a proof of $\frac{1}{y} \cdot \frac{1}{x} = \frac{1}{y \cdot x}$ C) $1 \cdot 1 \cdot \frac{1}{y \cdot x}$
D) $\frac{1}{y} \cdot \frac{1}{x} \cdot \left(y \cdot x \cdot \frac{1}{y \cdot x}\right)$ E) $\frac{1}{y} \cdot \frac{1}{x} \cdot 1$ F) $\left(y \cdot \frac{1}{y}\right) \cdot \left(x \cdot \frac{1}{x}\right) \cdot \frac{1}{y \cdot x}$ G) none of these

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**12.**

Select the statement/s that can be justified by [MBT]
A) $\frac{1}{3} \cdot \frac{1}{-6} = \frac{1}{3 \cdot -6}$ B) $\frac{1}{3} + \frac{1}{-6} = \frac{1+1}{3+-6}$ C) $\frac{1}{y \cdot x} = \frac{1}{y} \cdot \frac{1}{x}$ D) $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3 \cdot 2}$ E) none of these

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**13.**

Select all expression/s equivalent to:

$$\frac{1}{g} \cdot \frac{1}{t}$$

A) $\frac{1}{g} \cdot \frac{1}{t} \cdot \left(g \cdot t \cdot \frac{1}{g \cdot t}\right)$ B) $\left(g \cdot \frac{1}{g}\right) \cdot \left(t \cdot \frac{1}{t}\right) \cdot \frac{1}{g \cdot t}$ C) $1 \cdot 1 \cdot \frac{1}{g \cdot t}$ D) $\frac{1}{g} \cdot \frac{1}{t} \cdot 1$ E) $\frac{1}{g \cdot t}$
F) these choices outline a proof of $\frac{1}{g} \cdot \frac{1}{t} = \frac{1}{g \cdot t}$ G) none of these

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**14.**

Select all expression/s equivalent to:

$$\frac{1}{h} \cdot \frac{1}{u}$$

A) $\frac{1}{h \cdot u}$ B) $\left(h \cdot \frac{1}{h}\right) \cdot \left(u \cdot \frac{1}{u}\right) \cdot \frac{1}{h \cdot u}$ C) $\frac{1}{h} \cdot \frac{1}{u} \cdot 1$ D) $\frac{1}{h} \cdot \frac{1}{u} \cdot \left(h \cdot u \cdot \frac{1}{h \cdot u}\right)$ E) these choices outline a proof of $\frac{1}{h} \cdot \frac{1}{u} = \frac{1}{h \cdot u}$ F) $1 \cdot 1 \cdot \frac{1}{h \cdot u}$ G) none of these

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**15.**

Select the statement/s that can be justified by [MBT]
A) $\frac{1}{g \cdot u} = \frac{1}{g} \cdot \frac{1}{u}$ B) $\frac{1}{10} \cdot \frac{1}{-12} = \frac{1}{10 \cdot -12}$ C) $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{6 \cdot 2}$ D) $\frac{1}{10} + \frac{1}{-12} = \frac{1+1}{10+-12}$
E) none of these

---

## 3.3 Introduction to Primes and GCD

**Definition** For any integer or polynomial, $a$ we will use the verb to *factor* $a$, as meaning to take such integer and re-write as a product of two or more integers or polynomials. Said differently, to factor $a$ means to break it up into multiples such as $a = b \cdot c$

**Example** As an example, 24 can be factored as

$$24 = 3 \cdot 8$$

The 3 and the 8, and in general the pieces that the number is broken down into are called *factors*. Similarly, we have the notion of braking up a number into a sum of smaller pieces. This is called *partitioning* of a number. For example, this is a partition of the number 24:

$$24 = 20 + 4$$

**Definition** For any integer, $a$ we will use the verb to *partition* $a$, as meaning to take such integer and re-write as a sum of two or more integers. Said differently, to partition $a$ means to break it up into a sum such as $a = b + c$

**Example** Give a portions of 24:

$$24 = 20 + 4$$

**Definition** A *trivial factorization* of a number $a$ is one where one of the factors is 1 or -1.

**Examples** Trivial factorizations of "13":

$$13 = 1 \cdot 13$$

$$13 = -13 \cdot -1$$

**Definition** A *prime number* is a whole number that admits *only* trivial factorizations.

Below is a list of some prime numbers. Try to fact any of these. You will find that only trivial factorizations are possible on each, thus they are each a prime number.

**Examples** Some examples of prime numbers.

$$2, 3, 5, 7, 13, 17, 19, 53$$

In general, it is not easy to determine if a large number is prime or not. In fact, factoring is generally very difficult. So difficult that there are many security systems that depend precisely on this fact, and breaking the security system amounts roughly to factoring a very large number into its prime pieces.

It was once said that every even natural number can be written as the sum of exactly two prime number. For example,

$$20 = 13 + 7$$

Some of the brightest minds have worked on verifying this, yet this simple question has eluded all who have tried it. This, the Goldbach Conjecture remains an open problem today inviting all to try and explain the deep mystery as to why every even natural number can be written as the sum of two primes.

Another open problem is the *twin prime conjecture* which simply asks are there infinite many twin primes? A pair of twin primes is a pair of prime numbers which only differ by 2, such as 5 and 7, both primes and differ by two. 11 and 13 is another pair of prime numbers and 41 and 43 are also twin primes. Proving that there are infinite many *primes* was established very elegantly by Euclid many many years ago. However, proving that there are infinite many *twin primes* has been attempted by some of the brightest minds that have walked amongst us, yet it remains an open problem yet to be solved.

**Definition** When a natural number is factored, and the factors are factored, until they can be factored no more, this is called the *prime factorization* of the natural number.

**Example** The prime factorization of the natural number 24:

$$\begin{aligned} 24 &= 4 \cdot 6 && \text{TT} \\ &= 2 \cdot 2 \cdot 2 \cdot 3 && \text{TT,ALM} \\ &= 2^3 \cdot 3 && \text{BI} \end{aligned}$$

OR

$$\begin{aligned} 24 &= 3 \cdot 8 && \text{TT} \\ &= 3 \cdot 4 \cdot 2 && \text{TT,ALM} \\ &= 3 \cdot 2 \cdot 2 \cdot 2 && \text{TT,ALM} \\ &= 2^3 \cdot 3 && \text{BI} \end{aligned}$$

**Theorem 3.3.1.** *The fundamental theorem of arithmetic says that every natural number admits a unique prime factorization.*

Suppose we go on and prime factorize 24, and note that there are many ways to go about this. You could start by factoring as $24 = 4 \cdot 6$, or you could start by factoring as $24 = 3 \cdot 8$. The natural question that arises is wether or not, at the very end, you would get the same prime factorization. If we explore this idea just bit more with 24, on the first factorization we would get:

$$\begin{array}{rcll} 24 & = & 4 \cdot 6 & \text{TT} \\ & = & 2 \cdot 2 \cdot 2 \cdot 3 & \text{BI} \end{array}$$

While factoring the second way we get:

$$\begin{array}{rcll} 24 & = & 3 \cdot 8 & \text{TT} \\ & = & 3 \cdot 4 \cdot 2 & \text{BI} \\ & = & 3 \cdot 2 \cdot 2 \cdot 2 & \text{BI} \end{array}$$

Inspecting these two results shows that, other than the ordering, these two prime factorizations of 24 result in identical prime pieces, each has three 2's and one 3. The Fundamental Theorem of Arithmetic says this is no accident. In fact, it says that every natural number has a unique prime factorization, meaning other than the ordering, no matter how the factorization begins, all factorizations eventually lead to the same set of prime numbers. In some sense, the prime pieces are the DNA for the natural number.

It is often helpful to think about the act of prime factorizing a number as unmasking a number, or exposing the inner guts of a number, revealing its true composition, it's DNA code sequence. For example, the number 24 is in some way a mask, beneath the mask we see 24 is really made up of its key ingredients, a 3, a 2, another 2, and another 2. That is it! It's DNA code is $3 \cdot 2 \cdot 2 \cdot 2$. Prime factorizing is the act of revealing DNA code of a number.

One of the main reasons for all the vocabulary and ideas here, is that we are trying to learn more about fractions, after all this is the chapter on *rational numbers*. Two big topics that must be discussed when discussing rational numbers are 1) simplifying rational numbers and 2) adding rational numbers. As it turns out, prime numbers and prime factorization provide key ingredients extremely useful in finding *greatest common divisors*, *least common multiples*, simplifying fractions, as well as in adding fraction.

Consider the fraction

$$\frac{24}{36}$$

Observe, we could re-write it as:

$$\frac{24}{36} = \frac{8 \cdot 3}{12 \cdot 3} \tag{TT}$$

$$= \frac{8}{12} \cdot \frac{3}{3} \tag{MAT}$$

$$= \frac{8}{12} \cdot 1 \tag{JOT}$$

$$= \frac{8}{12} \tag{MiD}$$

Thus we conclude that

$$\frac{24}{36} = \frac{8}{12}$$

We typically call $\frac{8}{12}$ a more simplified version of $\frac{24}{36}$. In essence, what we did is 'cancel' a common factor from the numerator and the denominator, such and act is what we call 'simplifying the fraction', or 'simplifying the rational number'. After a moment of thinking, we might let ourselves wonder, could we do some more simplifying?

Indeed,

$$\frac{8}{12} = \frac{4 \cdot 2}{4 \cdot 3} \tag{TT}$$

$$= \frac{4}{4} \cdot \frac{2}{3} \tag{MAT}$$

$$= 1 \cdot \frac{2}{3} \tag{JOT}$$

$$= \frac{2}{3} \tag{MiD}$$

Thus we conclude about our original fraction

$$\frac{24}{36} = \frac{2}{3}$$

When is is not possible to simplify more, we call this the reduced fraction, or reduced rational. The punchline is that this can be done much more gracefully using prime factorization, as this will expose all the prime pieces making it readily visible what the greatest common divisors is. Observe, if we begin with the prime factorization of each:

$$\frac{24}{36} = \frac{3 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 2 \cdot 2} \tag{TT}$$

We see the common prime pieces a 3, a 2, and another 2, thus the greatest common divisor between 24 and 36 is $3 \cdot 2 \cdot 2$ in short we write

$$\gcd(24, 36) = 3 \cdot 2 \cdot 2 = 12$$

Knowing this, we could proceed to simplify completely and in one graceful sequence of steps as such.

$$\frac{24}{36} = \frac{3 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 2 \cdot 2 \cdot 3} \tag{TT}$$

$$= \frac{3 \cdot 2 \cdot 2}{3 \cdot 2 \cdot 2} \cdot \frac{2}{3} \tag{MAT}$$

$$= 1 \cdot \frac{2}{3} \tag{JOT}$$

$$= \frac{2}{3} \tag{MiD}$$

The incredulous student may grumble at such ideas, all to simply simplify fractions such as

$$\frac{24}{36}$$

Behold, there is so much more we can do with these ideas. The exact same ideas carry great power as they can be used to simply even the most complex of fractions. Consider the possibilities. Notice once we prime factorize, we see that is common, the greatest common divisor, and by the second line we separate the gcd, so that we may break it off, using MAT then reduce it using JOT and MId.

$$\frac{12x^4y^3}{-16x^5y^2} = \frac{3\cdot 2\cdot 2\cdot xxxxyyy}{-1\cdot 2\cdot 2\cdot 2\cdot 2\cdot xxxxxyy} \qquad \text{(TT, NExpo, MT)}$$

$$= \frac{2\cdot 2\cdot xxxxyy\cdot 3y}{2\cdot 2\cdot xxxxyy\cdot -1\cdot 2\cdot 2\cdot x} \qquad \text{(CoLM, ALM)}$$

$$= \frac{2\cdot 2\cdot xxxxyy}{2\cdot 2\cdot xxxxyy}\cdot\frac{3y}{-1\cdot 2\cdot 2\cdot x} \qquad \text{(MAT)}$$

$$= 1\cdot\frac{3y}{-4x} \qquad \text{(JOT, NPT)}$$

$$= \frac{3y}{-4x} \qquad \text{(MiD)}$$

As a bonus we see the generalization of the gcd as a concept not just for natural numbers, but for anything we can prime factorize.

$$\gcd(12x^4y^3, -16x^5y^2) = 2\cdot 2\cdot xxxxyy = 4x^4y^2$$

### 3.3.1 Exercises

**1.**

Select the true statement/s about:

$$\frac{u^2xyz^3}{ux^2z}$$

A) $\frac{u^2xyz^3}{ux^2z} = \frac{u^2x^2yz^3}{u^2x^2yz^3}\cdot\frac{x}{uyz^2}$ B) $\frac{x}{uyz^2}$ is the reduced form of $\frac{u^2xyz^3}{ux^2z}$ C) none of these

**2.**

Select the true statement/s about:

$$\frac{u^2x^2y^2z^3}{u^3x^2z}$$

A) $\frac{u}{y^2z^2}$ is the reduced form of $\frac{u^2x^2y^2z^3}{u^3x^2z}$ B) $\frac{u^2x^2y^2z^3}{u^3x^2z} = \frac{u^3x^2y^2z^3}{u^3x^2y^2z^3} \cdot \frac{u}{y^2z^2}$
C) none of these

---

**3.**

Select the true statement/s about:

$$\frac{x}{ux^3}$$

A) $\frac{x}{ux^3} = \frac{x}{x} \cdot \frac{1}{ux^2}$ B) $\frac{1}{ux^2}$ is the reduced form of $\frac{x}{ux^3}$ C) none of these

---

**4.**

Select the true statement/s about:

$$\frac{3375xy^2}{250}$$

A) $\frac{27xy^2}{2}$ is the reduced form of $\frac{3375xy^2}{250}$ B) $\frac{3375xy^2}{250} = \frac{125}{125} \cdot \frac{27xy^2}{2}$ C) none of these

---

**5.**

Select the true statement/s about:

$$\frac{135x^2y}{15x^2}$$

A) $9y$ is the reduced form of $\frac{135x^2y}{15x^2}$ B) $\frac{135x^2y}{15x^2} = \frac{15x^2}{15x^2} \cdot \frac{9y}{1}$ C) none of these

---

**6.**

Select the true statement/s about:

$$10309$$

A) 10309 is prime B) The prime factorizations of 10309 is $10309 = 61 \cdot 169$
C) none of these

---

**7.**

Select the true statement/s about the gcd:

$$\gcd(uw^3xy^2z, u^2w^3xy^2z^3) =$$

A) $u^2xyz^2$ B) $uw^3xy^2z$ C) none of these

---

**8.**

Select the true statement/s about the gcd:

$$\gcd(18y^3, 675x^2y^3) =$$

A) $9y^3$ B) $1350x^2y^3$ C) none of these

---

**9.**

Select the true statement/s about the gcd:

$$\gcd(u^2w^3y^4z^2, u^3w^2x^3y^4) =$$

A) $u^2w^2y^4$ B) $u^4wx^3y^3z$ C) none of these

---

**10.**

Select the true statement/s about the gcd:

$$\gcd(u^4x^4z^3, u^3w^4x^3y^2z^3) =$$

A) $u^3x^3z^3$ B) $u^2wx^3yz^3$ C) none of these

---

**11.**

Select the true statement/s about:

$$\frac{x^3y^2z}{u^3x^3z}$$

A) $\frac{x^3y^2z}{u^3x^3z} = \frac{u^3x^3y^2z}{u^3x^3y^2z} \cdot \frac{u^3}{y^2}$ B) $\frac{u^3}{y^2}$ is the reduced form of $\frac{x^3y^2z}{u^3x^3z}$ C) none of these

---

**12.**

Select the true statement/s about:

$$4697$$

A) The prime factorizations of 4697 is $4697 = 61 \cdot 77$ B) 4697 is prime
C) none of these

---

**13.**

Select the true statement/s about the gcd:

$$\gcd(600x^3y^2, 25xy) =$$

A) $25xy$ B) $600x^3y^2$ C) none of these

---

**14.**

Select the true statement/s about:

$$\frac{u^3yz}{ux^2y^2z}$$

A) $\frac{u^3yz}{ux^2y^2z} = \frac{u^3x^2y^2z}{u^3x^2y^2z} \cdot \frac{x^2y}{u^2}$ B) $\frac{x^2y}{u^2}$ is the reduced form of $\frac{u^3yz}{ux^2y^2z}$ C) none of these

---

**15.**

Select the true statement/s about:

$$\frac{ux^3yz^3}{u^2x}$$

A) $\frac{ux^3yz^3}{u^2x} = \frac{u^2x^3yz^3}{u^2x^3yz^3} \cdot \frac{u}{x^2yz^3}$ B) $\frac{u}{x^2yz^3}$ is the reduced form of $\frac{ux^3yz^3}{u^2x}$ C) none of these

---

**16.**

Select the true statement/s about the gcd:

$$\gcd(20x^3y^3, 3375x^2y) =$$

A) $5x^2y$ B) $13500x^3y^3$ C) none of these

---

**17.**

Select the true statement/s about:

$$1987$$

A) $1987 = 1 \cdot 1987$ is a trivial factorization of 1987 B) 1987 is prime
C) none of these

---

**18.**

Select the true statement/s about the gcd:

$$\gcd(u^3w^4x^3y^2z^2, w^2x^3y^2z^2) =$$

A) $w^2x^3y^2z^2$ B) $u^4z^3$ C) none of these

---

**19.**

Select the true statement/s about the gcd:

$$\gcd(u^3w^4xyz^4, w^3x^3z^2) =$$

A) $u^4wz^4$ B) $w^3xz^2$ C) none of these

---

**20.**

Select the true statement/s about:

$$4693$$

A) 4693 is prime B) The prime factorizations of 4693 is $4693 = 19 \cdot 247$
C) none of these

---

## 3.4 Adding Fractions

In the last few sections, we learned extensive ways to multiply and simplify fractions. In this section, we consider the question of *adding fractions*. The discussion of primes and prime factorization will be very relevant in this section, as it was in the previous one.

First we begin by considering the addition of fractions where the denominator matches, such as

$$\frac{3}{10} + \frac{9}{10}$$

We are at the fortunate place now where we have already learned most of the definitions needed. In this case we need nothing else but the tools we already have. Observe:

$$\begin{aligned}
&\frac{3}{10} + \frac{9}{10} && \text{(given)} \\
=&3 \cdot \frac{1}{10} + 9 \cdot \frac{1}{10} && \text{(def } a/b\text{)}
\end{aligned}$$

At this point we can say, "it looks like a $\frac{1}{10}$ was distributed!" Thus, we can proceed as:

$$\begin{aligned}
&\frac{3}{10} + \frac{9}{10} && \text{(given)} \\
=&3 \cdot \frac{1}{10} + 9 \cdot \frac{1}{10} && \text{(def } a/b\text{)} \\
=&(3+9)\frac{1}{10} && \text{(DL)} \\
=&12 \cdot \frac{1}{10} && \text{(AT)} \\
=&\frac{12}{10} && \text{(def } a/b\text{)}
\end{aligned}$$

More generally,

$$\begin{aligned} &\frac{A}{C} + \frac{B}{C} && \text{(given)} \\ =&A \cdot \frac{1}{C} + B \cdot \frac{1}{C} && \text{(def } a/b\text{)} \\ =&(A+B) \cdot \frac{1}{C} && \text{(DL)} \\ =&\frac{A+B}{C} && \text{(def } a/b\text{)} \end{aligned}$$

We package it, rack it!

**Theorem 3.4.1.** *The Add the Tops Theorem [ATT]*

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

*as usual, so long as $A, B$ and $C$ are numbers*[1] *for which our axioms apply, and $C \neq 0$*

**Examples**

$$\frac{3}{5} + \frac{1}{5} = \frac{4}{5} \qquad \text{(ATT)}$$

$$\frac{1}{7} + \frac{3}{7} = \frac{4}{7} \qquad \text{(ATT)}$$

$$\frac{3}{x^2} + \frac{x}{x^2} = \frac{3+x}{x^2} \qquad \text{(ATT)}$$

$$\frac{3}{x+1} + \frac{x}{x+1} = \frac{3+x}{x+1} \qquad \text{(ATT)}$$

$$\frac{3+x}{y} = \frac{3}{y} + \frac{x}{y} \qquad \text{(ATT [backwards])}$$

[1] when applicable we will include the addition table with this theorem

ATT takes care of addition of fractions when the denominators match. Things get much more interesting when they do not match. For example, Consider adding

$$\frac{2}{3} + \frac{4}{5}$$

Since the denominators are different, this expression would not be a good candidate for the ATT theorem. Instead, what we want to do is tweak each fraction until the denominator *do* match. We will do this, roughly speaking, by "unsimplifying" the fractions. Recall in the last section we simplified or reduced fractions by eliminating common factors. In this case we want to undo that.. We want to multiply numerator and denominator by factors so that the denominators match. Observe...

$$\begin{aligned}
&\frac{2}{3} + \frac{4}{5} && \text{(given)}\\
=&\frac{2}{3} \cdot 1 + \frac{4}{5} \cdot 1 && \text{(MiD)}\\
=&\frac{2}{3} \cdot \frac{5}{5} + \frac{4}{5} \cdot \frac{3}{3} && \text{(JOT)}\\
=&\frac{10}{15} + \frac{12}{15} && \text{(JOT)}
\end{aligned}$$

Pause here and note, we have indeed unsimplified the fractions, instead of $\frac{2}{3}$ we have $\frac{10}{15}$ and instead of $\frac{4}{5}$ we have $\frac{12}{15}$. This accomplishes something very important, namely that we have a common denominator, thus we are now ready for ATT.

$$\begin{aligned}
&\frac{2}{3} + \frac{4}{5} && \text{(given)}\\
=&\frac{2}{3} \cdot 1 + \frac{4}{5} \cdot 1 && \text{(MiD)}\\
=&\frac{2}{3} \cdot \frac{5}{5} + \frac{4}{5} \cdot \frac{3}{3} && \text{(JOT)}\\
=&\frac{10}{15} + \frac{12}{15} && \text{(MAT)}\\
=&\frac{22}{15} && \text{(ATT)}
\end{aligned}$$

There is however one important element left to explain. How do we know what to multiply each fraction by so that the denominators match? One option may be to multiply the first one by $\frac{18}{8}$ and the second one by $\frac{12}{12}$ Then they would each have denominator of 216. This however, may not be the most efficient or preferred method. Although, beauty is concededly subjective, we propose a more graceful way to this is by prime-factorizing each of the denominators then you can simply see what each demonstrators is missing so that they both match. Consider our next example. Add

$$\frac{5}{12} + \frac{7}{18}$$

As is, figuring out what needs to be multiplied may be a mystery to most of us. Prime-factorizing exposes all the key elements.

$$\frac{5}{3 \cdot 2 \cdot 2} + \frac{7}{3 \cdot 3 \cdot 2}$$

If these are to match then they have to have the same number of 3's and the same number of 2's. We can accomplish by supplying one 3 to the first fraction and one 2 to the second fraction, this way they will both have exactly two 3s and two 2s, and alas the denominators will match.

$$\frac{5}{3 \cdot 2 \cdot 2} + \frac{7}{3 \cdot 3 \cdot 2} \tag{given}$$

$$=\frac{5}{3 \cdot 2 \cdot 2} \cdot 1 + \frac{7}{3 \cdot 3 \cdot 2} \cdot 1 \tag{MiD}$$

$$=\frac{5}{3 \cdot 2 \cdot 2} \cdot \frac{3}{3} + \frac{7}{3 \cdot 3 \cdot 2} \cdot \frac{2}{2} \tag{JOT}$$

$$=\frac{15}{3 \cdot 2 \cdot 2 \cdot 3} + \frac{14}{3 \cdot 3 \cdot 2 \cdot 2} \tag{MAT}$$

$$=\frac{15}{3 \cdot 3 \cdot 2 \cdot 2} + \frac{14}{3 \cdot 3 \cdot 2 \cdot 2} \tag{ALM, CoLM}$$

$$=\frac{15}{36} + \frac{14}{36} \tag{TT}$$

$$=\frac{29}{36} \tag{ATT}$$

This is perhaps a good place to introduce the term '*least common multiple [LCM]*'. the example above provides one way to understand the LCM(12,18) as the most efficient denominator for adding fractions with a denominator 12 and a denominator 18. Towards the end of this section we will ready to provide a more rigorous definition, for now let us continue to practice constructing LCMs which, roughly speaking, amounts to constructing least common denominators. Consider adding

$$\frac{7}{20} + \frac{-13}{24}$$

Again a good way to see the least common denominator is to expose the DNA of the denominators...

$$\frac{7}{5 \cdot 2 \cdot 2} + \frac{-13}{3 \cdot 2 \cdot 2 \cdot 2}$$

As we look at the denominators, we see both denominators will need one 5, three 2's, and one 3.

$$\frac{7}{5 \cdot 2 \cdot 2} + \frac{-13}{3 \cdot 2 \cdot 2 \cdot 2} \qquad \text{(given)}$$

$$= \frac{7}{5 \cdot 2 \cdot 2} \cdot 1 + \frac{-13}{3 \cdot 2 \cdot 2 \cdot 2} \cdot 1 \qquad \text{(MId)}$$

$$= \frac{7}{5 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 3}{2 \cdot 3} + \frac{-13}{3 \cdot 2 \cdot 2 \cdot 2} \cdot \frac{5}{5} \qquad \text{(JOT)}$$

$$= \frac{7 \cdot 2 \cdot 3}{5 \cdot 2 \cdot 2 \cdot 2 \cdot 3} + \frac{-13 \cdot 5}{3 \cdot 2 \cdot 2 \cdot 2 \cdot 5} \qquad \text{(MAT)}$$

$$= \frac{42}{120} + \frac{-65}{120} \qquad \text{(NPT, TT)}$$

$$= \frac{42 + -65}{120} \qquad \text{(ATT)}$$

$$= \frac{-23}{120} \qquad \text{(BI)}$$

Incidentally, this shows the $lcm(20, 24) = 120$. Moreover, this process works so well and consistent it can be used to add just about any and all fractions we will encounter in our class. Below, we consider a a slight more demanding example.

**Example** Add

$$\frac{3}{45x^3y^2} + \frac{2}{54x^2z^4}$$

A good call is to first prime-factorize.

$$\frac{3}{3 \cdot 3 \cdot 5 \cdot xxxyy} + \frac{2}{3 \cdot 3 \cdot 3 \cdot 2 \cdot xxzzzz}$$

We can go one prime piece at a time, looks at the 3's, we will need three of them on each fraction, then looking at the 5s we see we will one one 5 on each fraction, then one 2, three $x$'s, two $y$'s and four $z$'s.

$$\frac{3}{3 \cdot 3 \cdot 5 \cdot xxxyy} + \frac{2}{3 \cdot 3 \cdot 3 \cdot 2 \cdot xxzzzz} \tag{given}$$

$$= \frac{3}{3 \cdot 3 \cdot 5 \cdot xxxyy} \cdot 1 + \frac{2}{3 \cdot 3 \cdot 3 \cdot 2 \cdot xxzzzz} \cdot 1 \tag{MId}$$

$$= \frac{3}{3 \cdot 3 \cdot 5 \cdot xxxyy} \cdot \frac{3 \cdot 2zzzz}{3 \cdot 2zzzz} + \frac{2}{3 \cdot 3 \cdot 3 \cdot 2 \cdot xxzzzz} \cdot \frac{5xyy}{5xyy} \tag{JOT}$$

$$= \frac{3 \cdot 3 \cdot 2zzzz}{3 \cdot 3 \cdot 5 \cdot xxxyy \cdot 3 \cdot 2zzzz} + \frac{2 \cdot 5xyy}{3 \cdot 3 \cdot 3 \cdot 2 \cdot xxzzzz \cdot 5xyy} \tag{MAT}$$

$$= \frac{18z^4}{270x^3y^2z^4} + \frac{10xy^2}{270x^3y^2z^4} \tag{BI}$$

$$= \frac{18z^4 + 10xy^2}{270x^3y^2z^4} \tag{ATT}$$

At last, we can conclude this section with a slightly more powerful definition of GCD and LCM, and a fundamental connection between the two. Essentially, this is just a summary of our method of our findings of the

GCD's while reducing fractions, as well as our methods of finding LCMs in adding fractions.

Consider the $\gcd(x^3, x^2)$ in such a case, when we have a common prime piece, we define the

$$\gcd(x^3, x^2) = x^2$$

, that is, the prime number raised to the lower of the two exponents. On the other hand for the LCM we when a common prime is shared we define it to be that prime raised the higher of the two exponents. So,

$$lcm(x^3, x^2) = x^3$$

If there are no common primes in the prime factorization, such as $\gcd(x^3, y^2)$, then the gcd is 1 and the lcm is the product of the two.

$$\gcd(x^3, y^2) = 1 \qquad lcm(x^3, y^2) = x^3y^2$$

The same principle can be used to find gcd and lcm of just about anything once prime factorized. Again, the gcd is defined to be the product of the common primes, each raised to the lower of the two exponents, while the lcm is defined to be the product of all primes occurring, each raised to the higher of the two exponents.

From the previous example,

$$\gcd(45x^3y^2, 3^3 \cdot 2 \cdot x^2z^4)$$

can be written as:

$$\gcd(3^2 \cdot 5 \cdot x^3y^2, 3^3 \cdot 2 \cdot x^2z^4) = 3^{?}x^{?}$$

We see the common primes are 3 and $x$, just writting them down gets us on the right track. Now, we just have to look at the exponents to see how many 3's occur on each number. 3 occurs twice in $3^2 \cdot 5 \cdot x^3y^2$ and three times in $3^3 \cdot 2 \cdot x^2z^4$, the lower of the two numbers is 2. Similarly for $x$, the lower of the two exponents is 2, thus

$$\gcd(3^2 \cdot 5 \cdot x^3y^2, 3^3 \cdot 2 \cdot x^2z^4) = 3^{2}x^{2}$$

On the other hand for LCM, we proceed similarly, but choose *all the prime pieces*, and each raised to the maximum of corresponding exponents, thus

$$lcm(3^2 \cdot 5 \cdot x^3y^2, 3^3 \cdot 2 \cdot x^2z^4) = 5^{?}3^{?}2^{?}x^{?}y^{?}z^{?}$$

Now we just have to locate the higher of the exponents for each corresponding prime piece.

$$lcm(3^2 \cdot 5 \cdot x^3y^2, 3^3 \cdot 2 \cdot x^2z^4) = 5^1 3^3 2^1 x^3 y^3 z^4$$

### 3.4.1 Exercises

**1.** Determine the

$$lcm(w^3y^{14}, w^{12}y^9)$$

**2.** Compute the following sum, show all reasoning

$$\frac{-30}{y^3} + \frac{33}{y^4}$$

**3.** Determine the

$$lcm(u^{11}v^2, u^2v^{13})$$

**4.** Determine the

$$lcm(41, 41^{15})$$

**5.** Determine the

$$lcm(1, 37^{11})$$

**6.** Compute the following sum, show all reasoning

$$\frac{38}{q^5} + \frac{20}{q^5}$$

**7.** Compute the following sum, show all reasoning

$$\frac{-44}{3v^4} + \frac{53}{7v^6}$$

**8.** Determine the

$$\gcd(z^2r^{13}, q^{14}r)$$

**9.** Determine the

$$lcm(\alpha^{15}\mu^{13}, \alpha^2\mu^5)$$

**10.** Determine the

$$lcm(w^7r^7, v^{11}w^9)$$

**11.** Determine the

$$\gcd(\alpha^{13}\gamma^6, \alpha^3\gamma^{12})$$

**12.** Compute the following sum, show all reasoning

$$\frac{-16}{13w^5} + \frac{31}{11w^5}$$

**13.** Compute the following sum, show all reasoning

$$\frac{34}{3w^2} + \frac{17}{w^5}$$

**14.** Determine the

$$lcm(47^{10}, 47^{15})$$

**15.** Determine the

$$lcm(w^{13}q^{10}, t^{13}q^6)$$

**16.** Compute the following sum, show all reasoning

$$\frac{3}{2t^6} + \frac{4}{7t^4}$$

**17.** Determine the

$$\gcd(47^4, 47^7)$$

**18.** Compute the following sum, show all reasoning

$$\frac{33}{z^5} + \frac{6}{z^2}$$

**19.** Compute the following sum, show all reasoning

$$\frac{-3}{z^2} + \frac{3}{11z^7}$$

**20.** Compute the following sum, show all reasoning

$$\frac{46}{7q^5} + \frac{30}{q^5}$$

**21.** Compute the following sum, show all reasoning

$$\frac{21}{13w^7} + \frac{39}{w^2}$$

**22.** Compute the following sum, show all reasoning

$$\frac{-45}{v^6} + \frac{54}{v^6}$$

**23.** Compute the following sum, show all reasoning

$$\frac{53}{q^3v^8} + \frac{10}{qv^5}$$

**24.** Determine the

$$lcm(\alpha^4\beta, \alpha^{10}\beta^9)$$

**25.** Determine the

$$\gcd(59^6, 59^4)$$

**26.** Compute the following sum, show all reasoning

$$\frac{-7}{13t^3} + \frac{37}{5t}$$

**27.** Compute the following sum, show all reasoning

$$\frac{14}{z} + \frac{37}{z^6}$$

**28.** Determine the

$$\gcd(41^{13}, 41^9)$$

**29.** Determine the

$$\gcd(z^7u^5, t^4u^{14})$$

**30.** Determine the

$$\gcd(\alpha^{11}\phi^{15}, \alpha^4\phi^{11})$$

**31.**

Determine the

$$\gcd(\mu^3\omega^{12}, \mu^{10}\omega^2)$$

A) $\mu^3\omega^2$ B) $\mu^{10}\omega^{12}$ C) none of these

**32.**

Select the true statement/s about:

$$\frac{1}{11} + \frac{6}{1}$$

A) $= \frac{1}{11} \cdot 1 + \frac{6}{1} \cdot 1$ B) $= \frac{1 \cdot 1}{11} + \frac{6 \cdot 11}{11}$ C) $= \frac{1}{11} + \frac{6}{1}$ D) $= \frac{67}{11}$ E) $= \frac{1}{11} \cdot \frac{1}{1} + \frac{6}{1} \cdot \frac{11}{11}$
F) none of these

---

**33.**

Determine the

$$lcm(q^{14}u^{15}, q^5w^7)$$

A) $q^{14}u^{15}w^7$ B) $q^5$ C) none of these

---

**34.**

Select the true statement/s about:

$$\frac{39}{v} + \frac{49}{1}$$

A) $= \frac{39}{v} \cdot 1 + \frac{49}{1} \cdot 1$ B) $= \frac{39}{v} \cdot \frac{1}{1} + \frac{49}{1} \cdot \frac{v}{v}$ C) $= \frac{49v+39}{v}$ D) $= \frac{39}{v} + \frac{49}{1}$
E) $= \frac{39 \cdot 1}{v} + \frac{49 \cdot v}{v}$ F) none of these

---

**35.**

Select the true statement/s about:

$$\frac{1}{3} + \frac{4}{1}$$

A) $= \frac{1}{3} \cdot \frac{1}{1} + \frac{4}{1} \cdot \frac{3}{3}$ B) $= \frac{1}{3} + \frac{4}{1}$ C) $= \frac{13}{3}$ D) $= \frac{1 \cdot 1}{3} + \frac{4 \cdot 3}{3}$ E) $= \frac{1}{3} \cdot 1 + \frac{4}{1} \cdot 1$
F) none of these

---

## 3.5 Complex Fractions

In this section we define 'division' as a binary operation, and we introduce *complex fractions*. Thus far, we have worked with fractions of integers or variables. We want to now turn our attention to fractions of fractions. Fractions of fractions are sometimes called complex fractions. In this case, the name is misleading, they are not complicated nor complex. They are just stacks of fractions. They can usually be simplified to a regular fraction, and what we want to do is become comfortable with this process.

First the definition of *divide*. Division is to multiplication what subtraction is to addition. Recall we defined "$a - b$" as the sum $a$ plus the additive inverse of $b$. Similarly, we shall define division.

**Definition** *Definition of Divide [def ÷]* If $A$ and $B$ are numbers for which our axioms apply and $b \neq 0$, we define $A \div B$ as

$$\frac{A}{B}$$

**Examples**

$$15 \div 3 = \frac{15}{3} \qquad \text{(def } \div\text{)}$$

$$4 \div 7 = \frac{4}{7} \qquad \text{(def } \div\text{)}$$

$$(4 + x^2) \div (y^2 - 1) = \frac{4 + x^2}{y^2 - 1} \qquad \text{(def } \div\text{)}$$

$$10 \div \frac{3}{7} = \frac{10}{\frac{3}{7}} \qquad \text{(def } \div\text{)}$$

Note the definition is analogous to our definition of subtraction. Since $a \div b = \frac{a}{b} = a \cdot \frac{1}{b}$, this means $a \div b$ is defined as "$a$ times the multiplicative inverse of $b$, just as $a - b$ was defined as $a$ plus the additive inverse of $b$.

Moreover, the last entry on the examples above shows a fraction of fractions. These are commonly called *complex fractions*. Don't let the name persuade you to believe these are complicated or complex, they are not. The 'complex' part is is just in the name. To deal with these complex fractions, we have the very famous and useful theorem below.

**Theorem 3.5.1.** *Flip the Bottom Theorem [FBT] If $b$ and $c$ are non zero, then*

$$\frac{A}{\frac{b}{c}} = A \cdot \frac{c}{b}$$

*We leave it as an important exercise for the student to prove this theorem.*

**Examples**

$$\frac{15}{\frac{5}{6}} = 15 \cdot \frac{6}{5} \qquad \text{(FBT)}$$

$$\frac{12}{\frac{5x}{7y^2}} = 12 \cdot \frac{7y^2}{5x} \qquad \text{(FBT)}$$

$$\frac{12}{\frac{x+1}{\pi}} = 12 \cdot \frac{\pi}{x+1} \qquad \text{(FBT)}$$

$$5 \cdot \frac{1}{2} = \frac{5}{\frac{2}{1}} \qquad \text{(FBT (backwards))}$$

$$\frac{\frac{5}{13}}{\frac{x+1}{\pi}} = \frac{5}{13} \cdot \frac{\pi}{x+1} \qquad \text{(FBT)}$$

$$\frac{\frac{1}{3}}{\frac{x}{\pi}} = \frac{1}{3} \cdot \frac{\pi}{x} \qquad \text{(FBT)}$$

### 3.5.1 Exercises

**1.** Simplify the following expression:

$$\frac{\frac{14}{5}}{\frac{34}{26}}$$

**2.** Simplify the following expression:

$$\frac{5y^4}{2} \div \frac{13y^5}{14}$$

**3.** Simplify the following expression:

$$\frac{17}{\frac{4}{18}}$$

**4.** Simplify the following expression:

$$\frac{11w^2}{15} \div \frac{14w^5}{6}$$

**5.** Simplify the following expression:

$$\frac{3}{\frac{10}{15}}$$

**6.** Simplify the following expression:

$$\frac{8q^8}{\frac{7q^2}{12}}$$

**7.** Simplify the following expression:

$$\frac{\frac{w^8}{8}}{\frac{7w^4}{3}}$$

**8.** Simplify the following expression:

$$\frac{8u^4}{3} \div \frac{13u^8}{11}$$

**9.** Simplify the following expression:

$$\frac{\frac{18}{11}}{\frac{8}{17}}$$

**10.** Simplify the following expression:

$$\frac{13u^8}{10} \div \frac{6u^5}{7}$$

**11.** Simplify the following expression:

$$\frac{36}{\frac{17}{22}}$$

**12.** Simplify the following expression:

$$\frac{11w^7}{10} \div \frac{2w^9}{6}$$

**13.** Prove FBT [ie don't use it here]. Show the steps to demonstrate the following:

$$\frac{V}{\frac{v}{w}} = V \cdot \frac{w}{v}$$

**14.** Simplify the following expression:

$$\frac{9z^5}{\frac{3z^7}{10}}$$

**15.** Simplify the following expression:

$$\frac{33}{\frac{23}{22}}$$

**16.** Prove FBT [ie don't use it here]. Show the steps to demonstrate the following:

$$\frac{Y}{\frac{w}{y}} = Y \cdot \frac{y}{w}$$

**17.** Simplify the following expression:

$$\frac{11t^3}{12} \div \frac{9t^4}{5}$$

**18.** Simplify the following expression:

$$\frac{15v^6}{13} \div \frac{4v^5}{3}$$

**19.** Simplify the following expression:

$$\frac{4u^2}{7} \div \frac{u^5}{6}$$

**20.** Simplify the following expression:

$$\frac{7y^7}{9} \div \frac{y^8}{12}$$

**21.**

Select the statement/s that can be declared true by [FBT]:

A) $= \frac{Q}{\frac{u}{\lambda}} = Q \cdot \frac{\lambda}{u}$ B) $= \frac{U}{\frac{w}{\alpha}} = U \cdot \frac{w}{\alpha}$ C) $= \frac{X}{\frac{t}{\mu}} = X \cdot \frac{\mu}{t}$ D) $= \frac{T}{\frac{w}{\theta}} = T \div \frac{\theta}{w}$

E) $= \frac{R}{\frac{z}{\beta}} = R \div \frac{z}{\beta}$ F) none of these

---

**22.**

Select the statement/s that can be declared true by [FBT]:

A) $= \frac{Y}{\frac{w}{\theta}} = Y \cdot \frac{\theta}{w}$ B) $= \frac{W}{\frac{r}{\phi}} = W \div \frac{\phi}{r}$ C) $= \frac{U}{\frac{q}{\lambda}} = U \cdot \frac{q}{\lambda}$ D) $= \frac{X}{\frac{t}{\beta}} = X \div \frac{t}{\beta}$

E) $= \frac{R}{\frac{v}{\gamma}} = R \cdot \frac{\gamma}{v}$ F) none of these

---

**23.**

Select the statement/s that can be declared true by [FBT]:

A) $= \frac{Y}{\frac{w}{\theta}} = Y \cdot \frac{\theta}{w}$ B) $= \frac{Q}{\frac{r}{\phi}} = Q \div \frac{r}{\phi}$ C) $= \frac{W}{\frac{y}{\mu}} = W \div \frac{\mu}{y}$ D) $= \frac{V}{\frac{q}{\gamma}} = V \cdot \frac{q}{\gamma}$

E) $= \frac{U}{\frac{u}{\lambda}} = U \cdot \frac{\lambda}{u}$ F) none of these

---

**24.**

Select the statement/s that can be declared true by [FBT]:

A) $= \frac{X}{\frac{t}{\phi}} = X \div \frac{\phi}{t}$ B) $= \frac{V}{\frac{r}{\lambda}} = V \cdot \frac{\lambda}{r}$ C) $= \frac{U}{\frac{u}{\mu}} = U \div \frac{u}{\mu}$ D) $= \frac{T}{\frac{y}{\alpha}} = T \cdot \frac{\alpha}{y}$

E) $= \frac{R}{\frac{v}{\gamma}} = R \cdot \frac{v}{\gamma}$ F) none of these

---

**25.**

Select the statement/s that can be declared true by [FBT]:

A) $= \frac{R}{\frac{w}{\gamma}} = R \cdot \frac{\gamma}{w}$ B) $= \frac{V}{\frac{z}{\lambda}} = V \cdot \frac{\lambda}{z}$ C) $= \frac{X}{\frac{r}{\phi}} = X \div \frac{r}{\phi}$ D) $= \frac{Q}{\frac{q}{\mu}} = Q \cdot \frac{q}{\mu}$

E) $= \frac{W}{\frac{v}{\alpha}} = W \div \frac{\alpha}{v}$ F) none of these

---

## 3.6 Introduction to Linear Equations

We are now coming to a conclusion of our third chapter. We now have enough ideas, tools, and methods to begin solving equations. Solving equations will be an essential part of the remainder of this course. One way to see equations is to see them as riddles. When we talk about 'solving the equation' we are referring to the 'solving of the riddle'. A most basic example may go something like this:

> I've got a number concealed inside my pocket. I won't tell you what the number is, but will offer a hint. When it is multiplied by 2, the result is equal to 10. Can you guess what the number is?

Roughly speaking, that is the essence of what we mean when we say "solve the equation". The equations can also be stated more explicitly. For example, in the above riddle, if we give this mysterious number a name, like '$x$', then the riddle portion:

> ...*When it is multiplied by 2, the result is equal to 10...*

becomes

> ...*When $x$ is multiplied by 2, the result is equal to 10...*

Or simply:

$$2x = 10$$

Generally, to solve the equation, we want to use all our might, our focus, our axioms and theorems, to isolate the unknown number all by itself on one side, this will help what resolve what this unknown quantity is equal to. Here is one way to proceed.

$$\begin{aligned}
2x &= 10 && \text{(given)}\\
\frac{1}{2}(2x) &= \frac{1}{2}(10) && \text{(CLM)}\\
\left(\frac{1}{2}\cdot 2\right)x &= \frac{10}{2} && \text{(ALM, def a/b)}\\
1\cdot x &= \frac{10}{2} && \text{(MinV)}\\
x &= 5 && \text{(MiD, BI)}
\end{aligned}$$

Of course most readers probably guessed this number when the riddle was first proposed, which may lead us to wonder why go through all of this to solve it? The main reason is that we want to establish powerful methods that consistently solve equations no matter how difficult they get. This riddle was particularly easy, but they are not all like this. Consider the following riddle.

> I've got a number concealed inside my pocket. I won't tell you what the number is, but will offer a hint. When it is multiplied by 2 then add -3, then multiply by 5 minus 7 times it, the result is equal to 10 times it divided by -2. Can you guess what the number is?

As you can see, our 'common sense' will quickly prove insufficient for these little riddles, thus we want general methods that at consistently reliable in helping solve these equations, and a good place to start developing theses methods is with the easy equations. Equations [such as the one we did solve] that can be written as

$$Ax = B$$

are generally the easiest to solve, and they are called *linear equations*. Here are some examples of linear equations.

**Examples**

$$3x = 7 \qquad \text{(ex: linear equation)}$$
$$7x = 12 \qquad \text{(ex: linear equation)}$$
$$-11x = \pi \qquad \text{(ex: linear equation)}$$
$$\frac{-3}{5}x = 17 + \sqrt{3} \qquad \text{(ex: linear equation)}$$

**Examples** These are examples of non-linear equations

$$7x^2 + 3x + 2 = 0 \qquad \text{(ex: NON-linear equation)}$$
$$-11x^5 + 3x = \pi \qquad \text{(ex: NON-linear equation)}$$
$$\frac{-3}{5}x - \frac{1}{x} = 17 + \sqrt{3} \qquad \text{(ex: NON-linear equation)}$$
$$3^x = 7 \qquad \text{(ex: NON-linear equation)}$$

For the moment we will focus on the linear equations. The $A$ portion in $Ax = B$ is usually called the 'coefficient'. A working strategy for all of these is, as we did above, to simply kill the coefficient slapping a multiplicative coefficient killer [or multiplicative inverse] on each side. That usually cleans up nicely and isolated the variable, and we are done. Another example is in order.

**Example**

$$\begin{aligned} 5x &= \pi && \text{(given)} \\ \frac{1}{5}(5x) &= \frac{1}{5}(\pi) && \text{(CLM)} \\ \left(\frac{1}{5}\cdot 5\right)x &= \frac{\pi}{5} && \text{(ALM, def a/b)} \\ 1\cdot x &= \frac{\pi}{5} && \text{(MinV)} \\ x &= \frac{\pi}{5} && \text{(MiD, BI)} \end{aligned}$$

Because these steps are so similar we shall package them into a theorem.

**Theorem 3.6.1.** *kill the coefficient [KTC] if $A$ and $B$ are numbers for which our axioms apply, and $A \neq 0$ and*

$$Ax = B$$

*then*

$$x = \frac{B}{A}$$

Let us practice the [KTC].

**Examples**

**Solve**

$$5x = 20$$

solution:

$$\begin{aligned} 5x &= 20 && \text{(given)} \\ x &= \frac{20}{5} && \text{(KTC)} \\ x &= 4 && \text{(BI)} \end{aligned}$$

**Solve**

$$5x = 19$$

solution:

$$\begin{aligned} 5x &= 19 && \text{(given)} \\ x &= \frac{19}{5} && \text{(KTC)} \end{aligned}$$

**Solve**

$$\pi x = \sqrt{3}$$

solution:

$$\begin{aligned} \pi x &= \sqrt{3} && \text{(given)} \\ x &= \frac{\sqrt{3}}{\pi} && \text{(KTC)} \end{aligned}$$

**Solve**

$$(1 + \pi + 5)x = \sqrt{3} + e^2$$

solution:

$$\begin{aligned} (1 + \pi + 5)x &= \sqrt{3} + e^2 && \text{(given)} \\ x &= \frac{\sqrt{3} + e^2}{1 + \pi + 5} && \text{(KTC)} \end{aligned}$$

Next, we consider a slightly more interesting type of linear equation is one where more than one term with $x$ in it appears. For example, consider the equation $3x + \pi x = -6$ This is slightly different because it is not exactly the form $Ax = B$. However, we can make it look like this. The perfect medicine for it is to collect the $x$ terms together using DL. This will effectively turn it into a type of equation such as $Ax = B$. While this we will not call a theorem, we do observe the strategy is worth noting. We shall call this strategy *collect & kill*.

**Example Solve**

$$3x + \pi x = -6$$

solution:

$$
\begin{aligned}
3x + \pi x &= -6 && \text{(given)}\\
(3+\pi)x &= -6 && \text{(DL (the 'collect' step))}\\
x &= \frac{-6}{3+\pi} && \text{(KTC (the 'kill' step))}
\end{aligned}
$$

This strategy will generally work for all linear equations that can be solved. So long as all the terms with $x$'s are on one side and all the $x$-less terms are on the other side, we can always collect the $x$ terms together then *kill* the coefficient and viola! Another example:

**Example Solve**

$$x + \pi x + \frac{2}{-5}x = -6 + \sqrt{5}$$

solution:

$$
\begin{aligned}
x + \pi x + \frac{2}{-5}x &= -6 + \sqrt{5} && \text{(given)}\\
1 \cdot x + \pi x + \frac{2}{-5}x &= -6 + \sqrt{5} && \text{(MiD)}\\
\left(1 + \pi + \frac{2}{-5}\right)x &= -6 + \sqrt{5} && \text{(DL (the 'collect' step))}\\
x &= \frac{-6+\sqrt{5}}{1+\pi+\frac{2}{-5}} && \text{(KTC (the 'kill' step))}
\end{aligned}
$$

It is important to emphasize, if all $x$ terms are on one side and the $x$ less terms on the other side, the *collect & kill* strategy will swiftly save the day. Of course this leads to another question. What if not all the $x$-terms are not separated, $x$'s on one side and $x$-less terms on the other side of the equation? Such as

$$3x + 7 = \pi x + -5$$

The general idea here is to shuffle things around. All terms with $x$'s on one side and the $x$-less terms on the other side. This is usually done using the Cancellation Law of Addition, by slapping an additive killer on both sides these can be eliminated one side and their inverse appears on the other side.

$$
\begin{aligned}
3x + 7 &= \pi x + -5 && \text{(given)} \\
(3x + 7) + -\pi x &= (\pi x + -5) + -\pi x && \text{(CLA)} \\
-\pi x + 3x + 7 &= -5 && \text{(BI)} \\
(-\pi x + 3x + 7) + -7 &= (-5) + -7 && \text{(CLA)} \\
-\pi x + 3x &= -12 && \text{(BI)}
\end{aligned}
$$

Once we have shuffled terms into their corresponding side, we repeat the *collect & kill* strategy

$$
\begin{aligned}
&\quad ....(\textit{continued})... \\
-\pi x + 3x &= -12 \\
(-\pi + 3)x &= -12 && \text{(DL)} \\
x &= \frac{-12}{-\pi + 3} && \text{(KTC)}
\end{aligned}
$$

The important lesson to take from this is that the CLA helps shuffle terms around so that all terms with $x$'s go on one side and $x$-less terms on the other side. The last lesson here is to observe that the terms are not always shuffles as they may be trapped and mixed with both terms with $x$s and terms with no $x$'s. For example

$$3(x + 1) = \frac{\pi x + 4}{3}$$

In here, the '$x + 1$' portion is 'trapped' inside the parenthesis and has both a term with $x$ and a term with no $x$. To make matters more interesting the right hand side also shows an $x$ term trapped on the numerator. These terms need to be set free, so that we can shuffle them into the correct side. Thus, the last piece of our strategy. Before we *collect & kill* we could always *free & shuffle* the terms into the correct side. If we do this, there will be no linear equation under the sun that can resist against our powerful solving strategies. By first doing the *free & shuffle* then the *collect & kill* strategy, we can solve *any and all* real solvable linear equations.

**Example Solve**

$$3(x + 1) = \frac{\pi x + 4}{3}$$

solution:

$$\begin{aligned}
3(x+1) &= \frac{\pi x + 4}{3} && \text{(given)} \\
3x + 3 \cdot 1 &= \frac{\pi x}{3} + \frac{4}{3} && \text{(DL, ATT)}
\end{aligned}$$

Now, the terms are free, we can now shuffle them into the right side.

$$\begin{aligned}
3(x+1) &= \frac{\pi x + 4}{3} && \text{(given)} \\
3x + 3 \cdot 1 &= \frac{\pi x}{3} + \frac{4}{3} && \text{(DL, ATT)} \\
(3x+3) + -3 &= \left(\frac{\pi x}{3} + \frac{4}{3}\right) + -3 && \text{(CLA, BI)} \\
3x &= \frac{\pi x}{3} + \frac{4}{3} + -3 && \text{(BI)} \\
(3x) + -\frac{\pi x}{3} &= \left(\frac{\pi x}{3} + \frac{4}{3} + -3\right) + -\frac{\pi x}{3} && \text{(CLA)} \\
3x + -\frac{\pi x}{3} &= \frac{4}{3} + -3 && \text{(BI)} \\
\left(3 + -\frac{\pi}{3}\right) x &= \frac{4}{3} + -3 && \text{(DL, BI)} \\
x &= \frac{\frac{4}{3} + -3}{3 + -\frac{\pi}{3}} && \text{(KTC)}
\end{aligned}$$

That's all the tools needed to solve all the linear equations inv6 the universe. It is not the only way to solve them, but it is one strategy that always works so long as the equation is linear and solvable and the numbers involved are ones to which our axioms apply, then the *free & shuffle* then *collect & kill* strategy will work.

### 3.6.1 Exercises

**1.** Solve for $v$ in the following equation:

$$14 \cdot v = 12$$

**2.** Solve for $q$ in the following equation:

$$\beta q + \alpha q = \alpha + 2 + \theta q$$

**3.** Solve for $z$ in the following equation:

$$(10 + 8)\, z = \frac{3}{10} + 6$$

**4.** Solve for $y$ in the following equation:

$$\pi y + \omega y + \beta y = \omega + 10$$

**5.** Solve for $q$ in the following equation:

$$(15 + \pi)\, q = \omega + 20$$

**6.** Solve for $z$ in the following equation:

$$\frac{\gamma z + \beta}{\theta} = \beta + 7 + \pi(z + \omega)$$

**7.** Solve for $t$ in the following equation:

$$(\beta + 6)\, t = \lambda + 12$$

**8.** Solve for $t$ in the following equation:

$$\frac{10t + 2}{\alpha} = \frac{9}{2} + 3(t + 6)$$

**9.** Solve for $u$ in the following equation:

$$\beta u + \pi u = 8 + \pi + \phi u$$

**10.** Solve for $w$ in the following equation:

$$\frac{5}{7}w + \frac{3}{4}w + \frac{9}{10} = \frac{3}{4} + 9 + \frac{8}{11}w$$

**11.** Solve for $w$ in the following equation:

$$\frac{\phi w + \pi}{\beta} = 8 + \pi + \alpha(w + \omega)$$

**12.** Solve for $v$ in the following equation:

$$1v + \frac{11}{10}v + \frac{9}{5} = \frac{11}{10} + 7 + \frac{6}{7}v$$

**13.** Solve for $t$ in the following equation:

$$\mu t + \alpha t + \theta t = \alpha + 8$$

**14.** Solve for $v$ in the following equation:

$$\theta v + \mu v = \mu + 6 + \phi v$$

**15.** Solve for $t$ in the following equation:

$$\frac{\alpha t + \lambda}{\pi} = \lambda + 10 + \theta(t + \mu)$$

**16.** Solve for $y$ in the following equation:

$$\frac{\pi y + \gamma}{\theta} = \gamma + 8 + \alpha(y + \mu)$$

**17.** Solve for $y$ in the following equation:

$$\frac{1}{3}y + \frac{4}{11}y + 6y = \frac{4}{11} + 3$$

**18.** Solve for $t$ in the following equation:

$$\frac{6t + 5}{\alpha} = 7 + 7(t + 8)$$

**19.** Solve for $u$ in the following equation:

$$\pi u + \lambda u + \theta u = \lambda + 10$$

**20.** Solve for $z$ in the following equation:

$$\frac{9}{10}z + 10z + \frac{7}{2} = 10 + 5 + \frac{8}{3}z$$

## 3.7 Chapter 3 Review

ESSENTIAL DEFINITIONS & THEOREMS

| *Definition/Theorem* | Short | Example |
|---|---|---|
| *Add Tops Theorem* | $[ATT]$ | $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$ |
| *Just One Theorem* | $[JOT]$ | $\frac{7}{7}=1$ |
| *Multiply Acorss Theorem* | $[MAT]$ | $\frac{2}{3}\cdot\frac{4}{5}=\frac{8}{15}$ |
| *Multiply Bottoms Theorem* | $[MBT]$ | $\frac{1}{3}\cdot\frac{1}{5}=\frac{1}{15}$ |
| *Negative Wherever you Want* | $[NWW]$ | $\frac{-2}{3}=-\frac{2}{3}=\frac{2}{-3}$ |
| *One Under Theorem* | $[OUT]$ | $5=\frac{5}{1}$ |
| *Definition of Divide* | $[Def\div]$ | $5\div 6=\frac{5}{6}$ |
| *Flip Bottoms Theorem* | $[FBT]$ | $\frac{7}{\frac{3}{5}}=7\cdot\frac{5}{3}$ |
| *Kill The Coeffificent* | $[KTC]$ | $Ax=B \implies x=\frac{B}{A}$ |

# Some Answers

## Section 3.1

**1.** BCEFG

**2.** BCD

**3.** CD

**4.** AB

**5.** BCEFI

**6.** ABD

**7.** ABD

**8.** ABDFI

**9.** AC

**10.** BDEFG

**11.** BD

**12.** AFGHI

**13.** BCD

**14.** ABD

**15.** BC

**16.** answers can vary

## Section 3.2

**1.** CE

**2.** ABCDEF

**3.** ABCDEF

**4.** CE

**5.** ABCDEF

**6.** ABCDF

**7.** ABCDEF

**8.** ABCDEF

**9.** ABCDEF

**10.** ABCDEF

**11.** ABCDEF

**12.** ACD

**13.** ABCDEF

**14.** ABCDEF

**15.** ABC

## Section 3.3

**1.** C

**2.** C

**3.** AB

**4.** AB

**5.** AB

**6.** C

**7.** B

**8.** A

**9.** A

**10.** A

**11.** C

**12.** C

**13.** A

**14.** C

**15.** C

**16.** A

**17.** AB

**18.** A

**19.** B

**20.** C

## Section 3.4

**1.** $lcm(w^3y^{14}, w^{12}y^9) = w^{12}y^{14}$

**2.**

$$
\begin{aligned}
&\frac{-30}{y^3} + \frac{33}{y^4} && \text{(given)}\\
=&\frac{-30}{y^3} \cdot 1 + \frac{33}{y^4} \cdot 1 && \text{(MiD)}\\
=&\frac{-30}{y^3} \cdot \frac{y}{y} + \frac{33}{y^4} \cdot \frac{1}{1} && \text{(JOT)}\\
=&\frac{-30 \cdot y}{y^4} + \frac{33 \cdot 1}{y^4} && \text{(MAT, BI)}\\
=&\frac{-30y}{y^4} + \frac{33}{y^4} && \text{(BI)}\\
=&\frac{-30y + 33}{y^4} && \text{(ATT)}\\
=&\frac{33 - 30y}{y^4} && \text{(ATT)}
\end{aligned}
$$

**3.** $lcm(u^{11}v^2, u^2v^{13}) = u^{11}v^{13}$

**4.** $lcm(41, 41^{15}) = 41^{15}$

**5.** $lcm(1, 37^{11}) = 37^{11}$

**6.**

$$\begin{aligned}
&\frac{38}{q^5} + \frac{20}{q^5} && \text{(given)}\\
=&\frac{38}{q^5} \cdot 1 + \frac{20}{q^5} \cdot 1 && \text{(MiD)}\\
=&\frac{38}{q^5} \cdot \frac{1}{1} + \frac{20}{q^5} \cdot \frac{1}{1} && \text{(JOT)}\\
=&\frac{38 \cdot 1}{q^5} + \frac{20 \cdot 1}{q^5} && \text{(MAT, BI)}\\
=&\frac{38}{q^5} + \frac{20}{q^5} && \text{(BI)}\\
=&\frac{38 + 20}{q^5} && \text{(ATT)}\\
=&\frac{58}{q^5} && \text{(ATT)}
\end{aligned}$$

**7.**

$$\begin{aligned}
&\frac{-44}{3v^4} + \frac{53}{7v^6} && \text{(given)}\\
=&\frac{-44}{3v^4} \cdot 1 + \frac{53}{7v^6} \cdot 1 && \text{(MiD)}\\
=&\frac{-44}{3v^4} \cdot \frac{7v^2}{7v^2} + \frac{53}{7v^6} \cdot \frac{3}{3} && \text{(JOT)}\\
=&\frac{-44 \cdot 7v^2}{21v^6} + \frac{53 \cdot 3}{21v^6} && \text{(MAT, BI)}\\
=&\frac{-308v^2}{21v^6} + \frac{159}{21v^6} && \text{(BI)}\\
=&\frac{-308v^2 + 159}{21v^6} && \text{(ATT)}\\
=&\frac{159 - 308v^2}{21v^6} && \text{(ATT)}
\end{aligned}$$

**8.** $\gcd(z^2r^{13}, q^{14}r) = r$

**9.** $lcm(\alpha^{15}\mu^{13}, \alpha^2\mu^5) = \alpha^{15}\mu^{13}$

**10.** $lcm(w^7r^7, v^{11}w^9) = v^{11}w^9r^7$

**11.** $\gcd(\alpha^{13}\gamma^6, \alpha^3\gamma^{12}) = \alpha^3\gamma^6$

**12.**

$$
\begin{aligned}
&\frac{-16}{13w^5} + \frac{31}{11w^5} && \text{(given)}\\
&= \frac{-16}{13w^5} \cdot 1 + \frac{31}{11w^5} \cdot 1 && \text{(MiD)}\\
&= \frac{-16}{13w^5} \cdot \frac{11}{11} + \frac{31}{11w^5} \cdot \frac{13}{13} && \text{(JOT)}\\
&= \frac{-16 \cdot 11}{143w^5} + \frac{31 \cdot 13}{143w^5} && \text{(MAT, BI)}\\
&= \frac{-176}{143w^5} + \frac{403}{143w^5} && \text{(BI)}\\
&= \frac{-176 + 403}{143w^5} && \text{(ATT)}\\
&= \frac{227}{143w^5} && \text{(ATT)}
\end{aligned}
$$

**13.**

$$
\begin{aligned}
&\frac{34}{3w^2} + \frac{17}{w^5} && \text{(given)}\\
=&\frac{34}{3w^2} \cdot 1 + \frac{17}{w^5} \cdot 1 && \text{(MiD)}\\
=&\frac{34}{3w^2} \cdot \frac{w^3}{w^3} + \frac{17}{w^5} \cdot \frac{3}{3} && \text{(JOT)}\\
=&\frac{34 \cdot w^3}{3w^5} + \frac{17 \cdot 3}{3w^5} && \text{(MAT, BI)}\\
=&\frac{34w^3}{3w^5} + \frac{51}{3w^5} && \text{(BI)}\\
=&\frac{34w^3 + 51}{3w^5} && \text{(ATT)}\\
=&\frac{34w^3 + 51}{3w^5} && \text{(ATT)}
\end{aligned}
$$

**14.** $lcm(47^{10}, 47^{15}) = 47^{15}$

**15.** $lcm(w^{13}q^{10}, t^{13}q^{6}) = w^{13}t^{13}q^{10}$

**16.**

$$
\begin{aligned}
& \frac{3}{2t^6} + \frac{4}{7t^4} && \text{(given)} \\
=& \frac{3}{2t^6} \cdot 1 + \frac{4}{7t^4} \cdot 1 && \text{(MiD)} \\
=& \frac{3}{2t^6} \cdot \frac{7}{7} + \frac{4}{7t^4} \cdot \frac{2t^2}{2t^2} && \text{(JOT)} \\
=& \frac{3 \cdot 7}{14t^6} + \frac{4 \cdot 2t^2}{14t^6} && \text{(MAT, BI)} \\
=& \frac{21}{14t^6} + \frac{8t^2}{14t^6} && \text{(BI)} \\
=& \frac{21 + 8t^2}{14t^6} && \text{(ATT)} \\
=& \frac{8t^2 + 21}{14t^6} && \text{(ATT)}
\end{aligned}
$$

**17.** $\gcd(47^4, 47^7) = 47^4$

**18.**

$$\frac{33}{z^5} + \frac{6}{z^2} \qquad \text{(given)}$$

$$= \frac{33}{z^5} \cdot 1 + \frac{6}{z^2} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{33}{z^5} \cdot \frac{1}{1} + \frac{6}{z^2} \cdot \frac{z^3}{z^3} \qquad \text{(JOT)}$$

$$= \frac{33 \cdot 1}{z^5} + \frac{6 \cdot z^3}{z^5} \qquad \text{(MAT, BI)}$$

$$= \frac{33}{z^5} + \frac{6z^3}{z^5} \qquad \text{(BI)}$$

$$= \frac{33 + 6z^3}{z^5} \qquad \text{(ATT)}$$

$$= \frac{6z^3 + 33}{z^5} \qquad \text{(ATT)}$$

**19.**

$$\frac{-3}{z^2} + \frac{3}{11z^7} \qquad \text{(given)}$$

$$= \frac{-3}{z^2} \cdot 1 + \frac{3}{11z^7} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{-3}{z^2} \cdot \frac{11z^5}{11z^5} + \frac{3}{11z^7} \cdot \frac{1}{1} \qquad \text{(JOT)}$$

$$= \frac{-3 \cdot 11z^5}{11z^7} + \frac{3 \cdot 1}{11z^7} \qquad \text{(MAT, BI)}$$

$$= \frac{-33z^5}{11z^7} + \frac{3}{11z^7} \qquad \text{(BI)}$$

$$= \frac{-33z^5 + 3}{11z^7} \qquad \text{(ATT)}$$

$$= \frac{3 - 33z^5}{11z^7} \qquad \text{(ATT)}$$

**20.**

$$
\begin{aligned}
&\frac{46}{7q^5} + \frac{30}{q^5} && \text{(given)}\\
=&\frac{46}{7q^5} \cdot 1 + \frac{30}{q^5} \cdot 1 && \text{(MiD)}\\
=&\frac{46}{7q^5} \cdot \frac{1}{1} + \frac{30}{q^5} \cdot \frac{7}{7} && \text{(JOT)}\\
=&\frac{46 \cdot 1}{7q^5} + \frac{30 \cdot 7}{7q^5} && \text{(MAT, BI)}\\
=&\frac{46}{7q^5} + \frac{210}{7q^5} && \text{(BI)}\\
=&\frac{46 + 210}{7q^5} && \text{(ATT)}\\
=&\frac{256}{7q^5} && \text{(ATT)}
\end{aligned}
$$

**21.**

$$\frac{21}{13w^7} + \frac{39}{w^2} \qquad \text{(given)}$$

$$= \frac{21}{13w^7} \cdot 1 + \frac{39}{w^2} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{21}{13w^7} \cdot \frac{1}{1} + \frac{39}{w^2} \cdot \frac{13w^5}{13w^5} \qquad \text{(JOT)}$$

$$= \frac{21 \cdot 1}{13w^7} + \frac{39 \cdot 13w^5}{13w^7} \qquad \text{(MAT, BI)}$$

$$= \frac{21}{13w^7} + \frac{507w^5}{13w^7} \qquad \text{(BI)}$$

$$= \frac{21 + 507w^5}{13w^7} \qquad \text{(ATT)}$$

$$= \frac{507w^5 + 21}{13w^7} \qquad \text{(ATT)}$$

**22.**

$$
\begin{aligned}
&\frac{-45}{v^6} + \frac{54}{v^6} && \text{(given)} \\
=&\frac{-45}{v^6} \cdot 1 + \frac{54}{v^6} \cdot 1 && \text{(MiD)} \\
=&\frac{-45}{v^6} \cdot \frac{1}{1} + \frac{54}{v^6} \cdot \frac{1}{1} && \text{(JOT)} \\
=&\frac{-45 \cdot 1}{v^6} + \frac{54 \cdot 1}{v^6} && \text{(MAT, BI)} \\
=&\frac{-45}{v^6} + \frac{54}{v^6} && \text{(BI)} \\
=&\frac{-45 + 54}{v^6} && \text{(ATT)} \\
=&\frac{9}{v^6} && \text{(ATT)}
\end{aligned}
$$

**23.**

$$\frac{53}{q^3v^8} + \frac{10}{qv^5} \qquad \text{(given)}$$

$$= \frac{53}{q^3v^8} \cdot 1 + \frac{10}{qv^5} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{53}{q^3v^8} \cdot \frac{1}{1} + \frac{10}{qv^5} \cdot \frac{q^2v^3}{q^2v^3} \qquad \text{(JOT)}$$

$$= \frac{53 \cdot 1}{q^3v^8} + \frac{10 \cdot q^2v^3}{q^3v^8} \qquad \text{(MAT, BI)}$$

$$= \frac{53}{q^3v^8} + \frac{10q^2v^3}{q^3v^8} \qquad \text{(BI)}$$

$$= \frac{53 + 10q^2v^3}{q^3v^8} \qquad \text{(ATT)}$$

$$= \frac{10q^2v^3 + 53}{q^3v^8} \qquad \text{(ATT)}$$

**24.** $lcm(\alpha^4\beta, \alpha^{10}\beta^9) = \alpha^{10}\beta^9$

**25.** $\gcd(59^6, 59^4) = 59^4$

**26.**

$$\frac{-7}{13t^3} + \frac{37}{5t} \qquad \text{(given)}$$

$$= \frac{-7}{13t^3} \cdot 1 + \frac{37}{5t} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{-7}{13t^3} \cdot \frac{5}{5} + \frac{37}{5t} \cdot \frac{13t^2}{13t^2} \qquad \text{(JOT)}$$

$$= \frac{-7 \cdot 5}{65t^3} + \frac{37 \cdot 13t^2}{65t^3} \qquad \text{(MAT, BI)}$$

$$= \frac{-35}{65t^3} + \frac{481t^2}{65t^3} \qquad \text{(BI)}$$

$$= \frac{-35 + 481t^2}{65t^3} \qquad \text{(ATT)}$$

$$= \frac{481t^2 - 35}{65t^3} \qquad \text{(ATT)}$$

**27.**

$$
\begin{aligned}
&\frac{14}{z}+\frac{37}{z^6} && \text{(given)}\\
=&\frac{14}{z}\cdot 1+\frac{37}{z^6}\cdot 1 && \text{(MiD)}\\
=&\frac{14}{z}\cdot\frac{z^5}{z^5}+\frac{37}{z^6}\cdot\frac{1}{1} && \text{(JOT)}\\
=&\frac{14\cdot z^5}{z^6}+\frac{37\cdot 1}{z^6} && \text{(MAT, BI)}\\
=&\frac{14z^5}{z^6}+\frac{37}{z^6} && \text{(BI)}\\
=&\frac{14z^5+37}{z^6} && \text{(ATT)}\\
=&\frac{14z^5+37}{z^6} && \text{(ATT)}
\end{aligned}
$$

**28.** $\gcd(41^{13}, 41^9) = 41^9$

**29.** $\gcd(z^7u^5, t^4u^{14}) = u^5$

**30.** $\gcd(\alpha^{11}\phi^{15}, \alpha^4\phi^{11}) = \alpha^4\phi^{11}$

**31.** A

**32.** ABCDE

**33.** A

**34.** ABCDE

**35.** ABCDE

# Section 3.5

**1.**

$$\frac{\frac{14}{5}}{\frac{34}{26}} \qquad \text{(given)}$$

$$= \frac{14}{5} \cdot \frac{26}{34} \qquad \text{(FBT)}$$

$$= \frac{182}{85} \qquad \text{(BI)}$$

**2.**

$$\frac{5y^4}{2} \div \frac{13y^5}{14} \qquad \text{(given)}$$

$$= \frac{\frac{5y^4}{2}}{\frac{13y^5}{14}} \qquad \text{(def } \div\text{)}$$

$$= \frac{5y^4}{2} \cdot \frac{14}{13y^5} \qquad \text{(FBT)}$$

$$= \frac{35}{13y} \qquad \text{(BI)}$$

**3.**

$$\frac{17}{\frac{4}{18}} \qquad \text{(given)}$$

$$= 17 \cdot \frac{18}{4} \qquad \text{(FBT)}$$

$$= \frac{17}{1} \cdot \frac{18}{4} \qquad \text{(OUT)}$$

$$= \frac{17 \cdot 18}{1 \cdot 4} \qquad \text{(MAT)}$$

$$= \frac{153}{2} \qquad \text{(BI)}$$

**4.**

$$\frac{11w^2}{15} \div \frac{14w^5}{6} \qquad \text{(given)}$$

$$= \frac{\frac{11w^2}{15}}{\frac{14w^5}{6}} \qquad \text{(def } \div\text{)}$$

$$= \frac{11w^2}{15} \cdot \frac{6}{14w^5} \qquad \text{(FBT)}$$

$$= \frac{11}{35w^3} \qquad \text{(BI)}$$

**5.**

$$\frac{3}{\frac{10}{15}} \qquad \text{(given)}$$

$$= 3 \cdot \frac{15}{10} \qquad \text{(FBT)}$$

$$= \frac{3}{1} \cdot \frac{15}{10} \qquad \text{(OUT)}$$

$$= \frac{3 \cdot 15}{1 \cdot 10} \qquad \text{(MAT)}$$

$$= \frac{9}{2} \qquad \text{(BI)}$$

**6.**

$$\frac{8q^8}{\frac{7q^2}{12}} \qquad \text{(given)}$$

$$= 8q^8 \cdot \frac{12}{7q^2} \qquad \text{(FBT)}$$

$$= \frac{96q^6}{7} \qquad \text{(BI)}$$

**7.**

$$\frac{\frac{w^8}{8}}{\frac{7w^4}{3}} \qquad \text{(given)}$$

$$= \frac{w^8}{8} \cdot \frac{3}{7w^4} \qquad \text{(FBT)}$$

$$= \frac{3w^4}{56} \qquad \text{(BI)}$$

**8.**

$$\frac{8u^4}{3} \div \frac{13u^8}{11} \qquad \text{(given)}$$

$$= \frac{\frac{8u^4}{3}}{\frac{13u^8}{11}} \qquad \text{(def } \div\text{)}$$

$$= \frac{8u^4}{3} \cdot \frac{11}{13u^8} \qquad \text{(FBT)}$$

$$= \frac{88}{39u^4} \qquad \text{(BI)}$$

**9.**

$$\frac{\frac{18}{11}}{\frac{8}{17}} \qquad \text{(given)}$$

$$= \frac{18}{11} \cdot \frac{17}{8} \qquad \text{(FBT)}$$

$$= \frac{153}{44} \qquad \text{(BI)}$$

**10.**

$$\frac{13u^8}{10} \div \frac{6u^5}{7} \qquad \text{(given)}$$

$$= \frac{\frac{13u^8}{10}}{\frac{6u^5}{7}} \qquad \text{(def } \div\text{)}$$

$$= \frac{13u^8}{10} \cdot \frac{7}{6u^5} \qquad \text{(FBT)}$$

$$= \frac{91u^3}{60} \qquad \text{(BI)}$$

**11.**

$$\begin{aligned} &\frac{36}{\frac{17}{22}} && \text{(given)} \\ =&36 \cdot \frac{22}{17} && \text{(FBT)} \\ =&\frac{36}{1} \cdot \frac{22}{17} && \text{(OUT)} \\ =&\frac{36 \cdot 22}{1 \cdot 17} && \text{(MAT)} \\ =&\frac{792}{17} && \text{(BI)} \end{aligned}$$

**12.**

$$\begin{aligned} &\frac{11w^7}{10} \div \frac{2w^9}{6} && \text{(given)} \\ =&\frac{\frac{11w^7}{10}}{\frac{2w^9}{6}} && \text{(def } \div\text{)} \\ =&\frac{11w^7}{10} \cdot \frac{6}{2w^9} && \text{(FBT)} \\ =&\frac{33}{10w^2} && \text{(BI)} \end{aligned}$$

**13.**

$$V \div \frac{v}{w} \qquad \text{(given)}$$

$$= \frac{V}{\frac{v}{w}} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{V}{\frac{v}{w}} \cdot \frac{\frac{w}{v}}{\frac{w}{v}} \qquad \text{(JOT)}$$

$$= \frac{V \cdot \frac{w}{v}}{\frac{v}{w} \cdot \frac{w}{v}} \qquad \text{(MAT)}$$

$$= \frac{V \cdot \frac{w}{v}}{\frac{v \cdot w}{w \cdot v}} \qquad \text{(MAT)}$$

$$= \frac{V \cdot \frac{w}{v}}{\frac{w \cdot v}{w \cdot v}} \qquad \text{(CoLM)}$$

$$= \frac{V \cdot \frac{w}{v}}{1} \qquad \text{(JOT)}$$

$$= V \cdot \frac{w}{v} \qquad \text{(OUT)}$$

**14.**

$$\frac{9z^5}{\frac{3z^7}{10}} \qquad \text{(given)}$$

$$= 9z^5 \cdot \frac{10}{3z^7} \qquad \text{(FBT)}$$

$$= \frac{30}{z^2} \qquad \text{(BI)}$$

**15.**

$$\frac{33}{\frac{23}{22}} \qquad \text{(given)}$$

$$= 33 \cdot \frac{22}{23} \qquad \text{(FBT)}$$

$$= \frac{33}{1} \cdot \frac{22}{23} \qquad \text{(OUT)}$$

$$= \frac{33 \cdot 22}{1 \cdot 23} \qquad \text{(MAT)}$$

$$= \frac{726}{23} \qquad \text{(BI)}$$

**16.**

$$Y \div \frac{w}{y} \qquad \text{(given)}$$

$$= \frac{Y}{\frac{w}{y}} \cdot 1 \qquad \text{(MiD)}$$

$$= \frac{Y}{\frac{w}{y}} \cdot \frac{\frac{y}{w}}{\frac{y}{w}} \qquad \text{(JOT)}$$

$$= \frac{Y \cdot \frac{y}{w}}{\frac{w}{y} \cdot \frac{y}{w}} \qquad \text{(MAT)}$$

$$= \frac{Y \cdot \frac{y}{w}}{\frac{w \cdot y}{y \cdot w}} \qquad \text{(MAT)}$$

$$= \frac{Y \cdot \frac{y}{w}}{\frac{y \cdot w}{y \cdot w}} \qquad \text{(CoLM)}$$

$$= \frac{Y \cdot \frac{y}{w}}{1} \qquad \text{(JOT)}$$

$$= Y \cdot \frac{y}{w} \qquad \text{(OUT)}$$

**17.**

$$\frac{11t^3}{12} \div \frac{9t^4}{5} \qquad \text{(given)}$$

$$= \frac{\frac{11t^3}{12}}{\frac{9t^4}{5}} \qquad \text{(def } \div\text{)}$$

$$= \frac{11t^3}{12} \cdot \frac{5}{9t^4} \qquad \text{(FBT)}$$

$$= \frac{55}{108t} \qquad \text{(BI)}$$

**18.**

$$\begin{aligned}
&\frac{15v^6}{13} \div \frac{4v^5}{3} && \text{(given)}\\
=&\frac{\frac{15v^6}{13}}{\frac{4v^5}{3}} && \text{(def } \div\text{)}\\
=&\frac{15v^6}{13} \cdot \frac{3}{4v^5} && \text{(FBT)}\\
=&\frac{45v}{52} && \text{(BI)}
\end{aligned}$$

**19.**

$$\begin{aligned}
&\frac{4u^2}{7} \div \frac{u^5}{6} && \text{(given)}\\
=&\frac{\frac{4u^2}{7}}{\frac{u^5}{6}} && \text{(def } \div\text{)}\\
=&\frac{4u^2}{7} \cdot \frac{6}{u^5} && \text{(FBT)}\\
=&\frac{24}{7u^3} && \text{(BI)}
\end{aligned}$$

**20.**

$$\frac{7y^7}{9} \div \frac{y^8}{12} \quad \text{(given)}$$

$$= \frac{\frac{7y^7}{9}}{\frac{y^8}{12}} \quad (\text{def } \div)$$

$$= \frac{7y^7}{9} \cdot \frac{12}{y^8} \quad \text{(FBT)}$$

$$= \frac{28}{3y} \quad \text{(BI)}$$

**21.** AC

**22.** AE

**23.** AE

**24.** BD

**25.** AB

# Section 3.6

**1.**

$$14v = 12 \quad \text{(given)}$$

$$v = \frac{12}{14} \quad \text{(KTC)}$$

**2.**

$$\begin{aligned}
\beta q + \alpha q + \mu &= \alpha + 2 + \theta q && \text{(given)}\\
(\beta q + \alpha q + \mu) + -\theta q &= (\alpha + 2 + \theta q) + -\theta q && \text{(CLA)}\\
\beta q + \alpha q + \mu + -\theta q &= \alpha + 2 && \text{(BI)}\\
(\beta q + \alpha q + \mu + -\theta q) + -\mu &= (\alpha + 2) + -\mu && \text{(CLA)}\\
\beta q + \alpha q + -\theta q &= \alpha + 2 + -\mu && \text{(BI)}\\
(\beta + \alpha + -\theta)\, q &= \alpha + 2 + -\mu && \text{(DL)}
\end{aligned}$$

$$q = \frac{\alpha + 2 + -\mu}{\alpha + \beta - \theta} \quad \text{(KTC)}$$

**3.**

$$\begin{aligned}(10+8)\,z &= \frac{3}{10}+6 && \text{(given)}\\ z &= \frac{\frac{3}{10}+6}{10+8} && \text{(KTC)}\\ z &= \frac{7}{20} && \text{(BI)}\end{aligned}$$

**4.**

$$\begin{aligned}\pi y+\omega y+\beta y &= \omega+10 && \text{(given)}\\ (\pi+\omega+\beta)\,y &= \omega+10 && \text{(DL)}\\ y &= \frac{\omega+10}{\beta+\omega+\pi} && \text{(KTC)}\end{aligned}$$

**5.**

$$\begin{aligned}(15+\pi)\,q &= \omega+20 && \text{(given)}\\ q &= \frac{\omega+20}{15+\pi} && \text{(KTC)}\end{aligned}$$

**6.**

$$\begin{aligned}\frac{\gamma z+\beta}{\theta} &= \beta+7+\pi(z+\omega) && \text{(given)}\\ \frac{\gamma z}{\theta}+\frac{\beta}{\theta} &= \beta+7+\pi z+\pi\omega && \text{(ATT, DL)}\\ \frac{\gamma z}{\theta}+-\pi z &= \beta+7+\pi\omega+-\frac{\beta}{\theta} && \text{(CLA, BI)}\\ \frac{\gamma}{\theta}\cdot z+-\pi\cdot z &= \beta+7+\pi\omega+-\frac{\beta}{\theta} && \text{(BI)}\\ \left(\frac{\gamma}{\theta}+-\pi\right)z &= \beta+7+\pi\omega+-\frac{\beta}{\theta} && \text{(DL)}\\ z &= \frac{\beta+7+\pi\omega+-\frac{\beta}{\theta}}{\frac{\gamma}{\theta}+-\pi} && \text{(KTC)}\end{aligned}$$

**7.**

$$\begin{aligned}(\beta+6)\,t &= \lambda+12 && \text{(given)}\\ t &= \frac{\lambda+12}{\beta+6} && \text{(KTC)}\end{aligned}$$

**8.**

$$
\begin{aligned}
\frac{10t+2}{\alpha} &= \frac{9}{2} + 3(t+6) && \text{(given)}\\
\frac{10t}{\alpha} + \frac{2}{\alpha} &= \frac{9}{2} + 3t + 36 && \text{(ATT, DL)}\\
\frac{10t}{\alpha} + -3t &= \frac{9}{2} + 36 + -\frac{2}{\alpha} && \text{(CLA, BI)}\\
\frac{10}{\alpha}\cdot t + -3\cdot t &= \frac{9}{2} + 36 + -\frac{2}{\alpha} && \text{(BI)}\\
\left(\frac{10}{\alpha} + -3\right) t &= \frac{9}{2} + 36 + -\frac{2}{\alpha} && \text{(DL)}\\
t &= \frac{\frac{9}{2} + 36 + -\frac{2}{\alpha}}{\frac{10}{\alpha} + -3} && \text{(KTC)}
\end{aligned}
$$

**9.**

$$
\begin{aligned}
\beta u + \pi u + \alpha &= 8 + \pi + \phi u && \text{(given)}\\
(\beta u + \pi u + \alpha) + -\phi u &= (8 + \pi + \phi u) + -\phi u && \text{(CLA)}\\
\beta u + \pi u + \alpha + -\phi u &= 8 + \pi && \text{(BI)}\\
(\beta u + \pi u + \alpha + -\phi u) + -\alpha &= (8 + \pi) + -\alpha && \text{(CLA)}\\
\beta u + \pi u + -\phi u &= 8 + \pi + -\alpha && \text{(BI)}\\
(\beta + \pi + -\phi)\, u &= 8 + \pi + -\alpha && \text{(DL)}\\
u &= \frac{8 + \pi + -\alpha}{\beta - \phi + \pi} && \text{(KTC)}
\end{aligned}
$$

**10.**

$$
\begin{aligned}
\frac{5}{7}w + \frac{3}{4}w + \frac{9}{10} &= \frac{3}{4} + 9 + \frac{8}{11}w && \text{(given)}\\
\left(\frac{5}{7}w + \frac{3}{4}w + \frac{9}{10}\right) + -\frac{8}{11}w &= \left(\frac{3}{4} + 9 + \frac{8}{11}w\right) + -\frac{8}{11}w && \text{(CLA)}\\
\frac{5}{7}w + \frac{3}{4}w + \frac{9}{10} + -\frac{8}{11}w &= \frac{3}{4} + 9 && \text{(BI)}\\
\left(\frac{5}{7}w + \frac{3}{4}w + \frac{9}{10} + -\frac{8}{11}w\right) + -\frac{9}{10} &= \left(\frac{3}{4} + 9\right) + -\frac{9}{10} && \text{(CLA)}\\
\frac{5}{7}w + \frac{3}{4}w + -\frac{8}{11}w &= \frac{3}{4} + 9 + -\frac{9}{10} && \text{(BI)}\\
\left(\frac{5}{7} + \frac{3}{4} + -\frac{8}{11}\right) w &= \frac{3}{4} + 9 + -\frac{9}{10} && \text{(DL)}\\
w &= \frac{\frac{3}{4} + 9 + -\frac{9}{10}}{-\frac{8}{11} + \frac{5}{7} + \frac{3}{4}} && \text{(KTC)}\\
w &= \frac{13629}{1135} && \text{(BI)}
\end{aligned}
$$

**11.**

$$\begin{aligned}
\frac{\phi w+\pi}{\beta} &= 8+\pi+\alpha(w+\omega) && \text{(given)}\\
\frac{\phi w}{\beta}+\frac{\pi}{\beta} &= 8+\pi+\alpha w+\alpha\omega && \text{(ATT, DL)}\\
\frac{\phi w}{\beta}+-\alpha w &= 8+\pi+\alpha\omega+-\frac{\pi}{\beta} && \text{(CLA, BI)}\\
\frac{\phi}{\beta}\cdot w+-\alpha\cdot w &= 8+\pi+\alpha\omega+-\frac{\pi}{\beta} && \text{(BI)}\\
\left(\frac{\phi}{\beta}+-\alpha\right)w &= 8+\pi+\alpha\omega+-\frac{\pi}{\beta} && \text{(DL)}\\
w &= \frac{8+\pi+\alpha\omega+-\frac{\pi}{\beta}}{\frac{\phi}{\beta}+-\alpha} && \text{(KTC)}
\end{aligned}$$

**12.**

$$\begin{aligned}
1v+\frac{11}{10}v+\frac{9}{5} &= \frac{11}{10}+7+\frac{6}{7}v && \text{(given)}\\
\left(1v+\frac{11}{10}v+\frac{9}{5}\right)+-\frac{6}{7}v &= \left(\frac{11}{10}+7+\frac{6}{7}v\right)+-\frac{6}{7}v && \text{(CLA)}\\
1v+\frac{11}{10}v+\frac{9}{5}+-\frac{6}{7}v &= \frac{11}{10}+7 && \text{(BI)}\\
\left(1v+\frac{11}{10}v+\frac{9}{5}+-\frac{6}{7}v\right)+-\frac{9}{5} &= \left(\frac{11}{10}+7\right)+-\frac{9}{5} && \text{(CLA)}\\
1v+\frac{11}{10}v+-\frac{6}{7}v &= \frac{11}{10}+7+-\frac{9}{5} && \text{(BI)}\\
\left(1+\frac{11}{10}+-\frac{6}{7}\right)v &= \frac{11}{10}+7+-\frac{9}{5} && \text{(DL)}
\end{aligned}$$

$$\begin{aligned}
v &= \frac{\frac{11}{10}+7+-\frac{9}{5}}{1+\frac{11}{10}-\frac{6}{7}} && \text{(KTC)}\\
v &= \frac{147}{29} && \text{(BI)}
\end{aligned}$$

**13.**

$$\begin{aligned}
\mu t+\alpha t+\theta t &= \alpha+8 && \text{(given)}\\
(\mu+\alpha+\theta)\,t &= \alpha+8 && \text{(DL)}
\end{aligned}$$

$$t = \frac{\alpha+8}{\alpha+\theta+\mu} \qquad \text{(KTC)}$$

**14.**

$$\begin{aligned}
\theta v + \mu v + \beta &= \mu + 6 + \phi v && \text{(given)}\\
(\theta v + \mu v + \beta) + -\phi v &= (\mu + 6 + \phi v) + -\phi v && \text{(CLA)}\\
\theta v + \mu v + \beta + -\phi v &= \mu + 6 && \text{(BI)}\\
(\theta v + \mu v + \beta + -\phi v) + -\beta &= (\mu + 6) + -\beta && \text{(CLA)}\\
\theta v + \mu v + -\phi v &= \mu + 6 + -\beta && \text{(BI)}\\
(\theta + \mu + -\phi)\, v &= \mu + 6 + -\beta && \text{(DL)}\\
v &= \frac{\mu + 6 + -\beta}{\theta + \mu - \phi} && \text{(KTC)}
\end{aligned}$$

**15.**

$$\begin{aligned}
\frac{\alpha t + \lambda}{\pi} &= \lambda + 10 + \theta(t + \mu) && \text{(given)}\\
\frac{\alpha t}{\pi} + \frac{\lambda}{\pi} &= \lambda + 10 + \theta t + \theta\mu && \text{(ATT, DL)}\\
\frac{\alpha t}{\pi} + -\theta t &= \lambda + 10 + \theta\mu + -\frac{\lambda}{\pi} && \text{(CLA, BI)}\\
\frac{\alpha}{\pi} \cdot t + -\theta \cdot t &= \lambda + 10 + \theta\mu + -\frac{\lambda}{\pi} && \text{(BI)}\\
\left(\frac{\alpha}{\pi} + -\theta\right) t &= \lambda + 10 + \theta\mu + -\frac{\lambda}{\pi} && \text{(DL)}\\
t &= \frac{\lambda + 10 + \theta\mu + -\frac{\lambda}{\pi}}{\frac{\alpha}{\pi} + -\theta} && \text{(KTC)}
\end{aligned}$$

**16.**

$$\begin{aligned}
\frac{\pi y + \gamma}{\theta} &= \gamma + 8 + \alpha(y + \mu) && \text{(given)}\\
\frac{\pi y}{\theta} + \frac{\gamma}{\theta} &= \gamma + 8 + \alpha y + \alpha\mu && \text{(ATT, DL)}\\
\frac{\pi y}{\theta} + -\alpha y &= \gamma + 8 + \alpha\mu + -\frac{\gamma}{\theta} && \text{(CLA, BI)}\\
\frac{\pi}{\theta} \cdot y + -\alpha \cdot y &= \gamma + 8 + \alpha\mu + -\frac{\gamma}{\theta} && \text{(BI)}\\
\left(\frac{\pi}{\theta} + -\alpha\right) y &= \gamma + 8 + \alpha\mu + -\frac{\gamma}{\theta} && \text{(DL)}\\
y &= \frac{\gamma + 8 + \alpha\mu + -\frac{\gamma}{\theta}}{\frac{\pi}{\theta} + -\alpha} && \text{(KTC)}
\end{aligned}$$

**17.**

$$\begin{aligned}
\frac{1}{3}y + \frac{4}{11}y + 6y &= \frac{4}{11} + 3 && \text{(given)}\\
\left(\frac{1}{3} + \frac{4}{11} + 6\right) y =& \frac{4}{11} + 3 && \text{(DL)}\\
y =& \frac{\frac{4}{11} + 3}{\frac{1}{3} + \frac{4}{11} + 6} && \text{(KTC)}\\
y =& \frac{111}{221} && \text{(BI)}
\end{aligned}$$

**18.**

$$\begin{aligned}
\frac{6t+5}{\alpha} &= 7 + 7(t+8) && \text{(given)}\\
\frac{6t}{\alpha} + \frac{5}{\alpha} &= 7 + 7t + 78 && \text{(ATT, DL)}\\
\frac{6t}{\alpha} + -7t &= 7 + 78 + -\frac{5}{\alpha} && \text{(CLA, BI)}\\
\frac{6}{\alpha} \cdot t + -7 \cdot t &= 7 + 78 + -\frac{5}{\alpha} && \text{(BI)}\\
\left(\frac{6}{\alpha} + -7\right) t &= 7 + 78 + -\frac{5}{\alpha} && \text{(DL)}\\
t &= \frac{7 + 78 + -\frac{5}{\alpha}}{\frac{6}{\alpha} + -7} && \text{(KTC)}
\end{aligned}$$

**19.**

$$\begin{aligned}
\pi u + \lambda u + \theta u &= \lambda + 10 && \text{(given)}\\
(\pi + \lambda + \theta)\, u =& \lambda + 10 && \text{(DL)}\\
u =& \frac{\lambda + 10}{\theta + \lambda + \pi} && \text{(KTC)}
\end{aligned}$$

**20.**

$$\begin{aligned}
\frac{9}{10}z + 10z + \frac{7}{2} &= 10 + 5 + \frac{8}{3}z && \text{(given)}\\
\left(\frac{9}{10}z + 10z + \frac{7}{2}\right) + -\frac{8}{3}z &= \left(10 + 5 + \frac{8}{3}z\right) + -\frac{8}{3}z && \text{(CLA)}\\
\frac{9}{10}z + 10z + \frac{7}{2} + -\frac{8}{3}z &= 10 + 5 && \text{(BI)}\\
\left(\frac{9}{10}z + 10z + \frac{7}{2} + -\frac{8}{3}z\right) + -\frac{7}{2} &= (10 + 5) + -\frac{7}{2} && \text{(CLA)}\\
\frac{9}{10}z + 10z + -\frac{8}{3}z &= 10 + 5 + -\frac{7}{2} && \text{(BI)}\\
\left(\frac{9}{10} + 10 + -\frac{8}{3}\right)z &= 10 + 5 + -\frac{7}{2} && \text{(DL)}
\end{aligned}$$

$$\begin{aligned}
z &= \frac{10 + 5 + -\frac{7}{2}}{-\frac{8}{3} + 10 + \frac{9}{10}} && \text{(KTC)}\\
z &= \frac{345}{247} && \text{(BI)}
\end{aligned}$$

# Bibliography

[1] Wikipedia Euclid's Elements Retrieved August 10, 2011

[2] Wikipedia Georg Cantor Retrieved August 2011

[3] Wikipedia Cantor's Diagonal Argument Retrieved August 2011

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