CONTENTS

CHAPTER 4

POLYNOMIALS

We concluded chapter 3 with a modest discussion of solving linear equations, a very important milestone. In this chapter, we expand on that or more accurately we begin sharpening our tools so that we can expand on that, and solve not just linear equations, but many other types of equations as well. Roughly speaking, this involves being comfortable when working with 'polynomials'. Recall, chapter 2 was essentially a chapter on integers, that is positive and negative numbers, and how to add, subtract, and multiply them. In chapter 3 we did the similar type of work but for fractions. A good way to think about this chapter is that polynomials are to this chapter, what rationals were to chapter 3, or what integers were to chapter 2. Once we get comfortable with polynomials, we will begin to solve much more interesting equations, than the [linear] ones we solved at the end of chapter 3.

4.1 Adding & Subtracting Polynomials

Let us begin by defining a 'monomial'.

Definition A *monomial in x with real coefficient* is defined as an expression that can be written as

$$Ax^n$$

Where A represents some constant real number, x is a variable, and n is a whole number. While x is called the *variable*, n is called the *degree of the term*, and A is called the coefficient. Another word for a *monomial* is a *term*.

We could very easily make variations of this definition. For example:

Definition A *monomial in x with integer coefficient* is defined as an expression that can be written as

$$Ax^n$$

Where A represents some integer number, x is a variable, and n is a whole number.

The *sum or difference* of two monomials is called a *binomial*, for example:

$$3x^2 + 5x$$

The sum of difference of three monomials is called a trinomial. Generally, we define,

Definition A *polynomial* is defined as the sum or difference of a collection of one or more monomials.

For example,

$$x^5 + 2x^3 - 5$$

is a polynomial. Often, it is convenient to *name* the polynomials. In the name we may often include some information about the polynomial, such as what is the variable. For example we could name the $x^5 + 2x^3 - 5$ as simply $p(x)$, meaning its name is p and the variable is identified as x. Thus we could write

$$p(x) = x^5 + 2x^3 - 5$$

Subsequently, we could just refer to $x^5 + 2x^3 - 5$ as simply $p(x)$, under the agreement that these are interchangeable. Meanwhile we could have a different polynomial with a different name, such as

$$h(x) = x^5 + x + 2$$

Then when we write $p(x)$ everyone should concur we are referring to $x^5 + 2x^3 - 5$ while if we mention $h(x)$ everyone should think of $x^5 + x + 2$

If we restrict the possible coefficients to *only integers*, and the possible variables to *only x's*, then we can call then entire set of such polynomials $\mathbf{Z}[x]$. Analogously, the set $\mathbf{Q}[t]$ represents all polynomials with *only rational* coefficients and only t variables, while $\mathbf{R}[x]$ represents the set of all polynomials in x with *real* coefficients. Moreover, we often abbreviate, instead of "$3x^2 + 5x + \sqrt{3}$ is a polynomial with real coefficients" we write:

$$3x^2 + 5x + \sqrt{3} \in \mathbf{R}[x]$$

where the symbol $\in$ is read as '*is an element of..*'. Thus.

$$3x^2 + 5x + \sqrt{3} \in \mathbf{R}[x]$$

can be read as "$3x^2 + 5x + \sqrt{3}$ is an element of the set of real polynomials"

Definition If $q(x)$ is a polynomial, we define the *degree of polynomial* $q(x)$ as the highest of the degrees of any of its terms. If $q(x)$ is a constant we say that its degree is zero.

Examples

$$4x^5 + 82x + 9 \qquad \text{(degree for this polynomial is 5)}$$

$$4x^9 + 27x + 9 \qquad \text{(degree for this polynomial is 9)}$$

$$5x^3 + 9 + 2^7 \qquad \text{(degree for this polynomial is 3)}$$

$$12 \qquad \text{(degree for this polynomial is 0)}$$

$$12 + x^2 + 5x \qquad \text{(degree for this polynomial is 2)}$$

Now that we have some of the basic vocabulary tackled, we are ready to start adding/subtracting polynomials.

Example Add

$$(3x^2 + 4x + 1) + (x^2 - 4x + 5)$$

solution:

$$
\begin{aligned}
&(3x^2 + 4x + 1) + (x^2 - 4x + 5) \\
&= (3x^2 + 1x^2) + (4x + -4x) + (1 + 5) && \text{(ALA, CoLA)} \\
&= (3 + 1)x^2 + (4 + -4)x + (1 + 5) && \text{(DL)} \\
&= 4x^2 + 0x + 6 && \text{(BI)} \\
&= 4x^2 + 0 + 6 && \text{(0MT)} \\
&= 4x^2 + 6 && \text{(AId)}
\end{aligned}
$$

It should be noted that based on the above example, it should be clear that the underlying principle in adding polynomials is to use ALA and CoLA to group associate like terms together, and *like terms* for us will mean same degree. That is, we collect degree two terms together, degree one terms together and so on. Then we can use DL to factor out the variable in effect just adding the coefficients. This should be reminiscent of adding natural numbers as you did in grade-school where you collect units, collect tens, and collect 100's more or less. Such as

$$
\begin{array}{r}
4\,8\,2\,5 \\
+ \quad 5\,3\,0\,7 \\
\hline
1\,0\,1\,3\,2
\end{array}
$$

Based on this idea and the the above comments, we can alternatively add polynomials the 'Kindergarden' [KG] way such as:

$$
\begin{array}{rr}
 & 3x^2 + 4x + 1 \\
+ & x^2 - 4x + 5 \\
\hline
 & 4x^2 + 6
\end{array}
\qquad \text{(KG)}
$$

Example Subtract

$$(3x^2 + 4x + 1) - (x^2 - 4x + 5)$$

solution:

$$\begin{aligned}
&(3x^2+4x+1)-(x^2-4x+5) \\
&=(3x^2+4x+1)+-1\cdot(x^2+-4x+5) && \text{(def a-b)} \\
&=(3x^2+4x+1)+-1x^2+-1\cdot-4x+-1\cdot 5 && \text{(DL)} \\
&=3x^2+4x+1+-1x^2+4x+-5 && \text{(DL)} \\
&=(3x^2+-1x^2)+(4x+4x)+(1+-5) && \text{(ALA, CoLA)} \\
&=(3+-1)x^2+(4+4)x+(1+-5) && \text{(DL)} \\
&=2x^2+8x+-4 && \text{(BI)}
\end{aligned}$$

OR we can try the KG method. .

$$\begin{array}{rr}
 & 3x^2+4x+1 \\
- & x^2-4x+5 \\
\hline
 & 2x^2+8x-4
\end{array} \qquad \text{(KG)}$$

4.1.1 Exercises

1. Compute and simplify:

$$\left(1-r^4\right)+\left(7r^6+2r^5+6r^4+5r^3+3r^2+2r+4\right)$$

2. Compute and simplify:

$$\begin{array}{rl}
 & 2q^3+4q+2 \\
+ & 2q^4-4q^3+q^2+7q \\
\hline
\end{array}$$

3. Compute and simplify:

$$\begin{array}{rl}
 & 4w^6+6w^5+7w^4+w^2-5w+3 \\
- & \left(-w^6+5w^5+5w^3+5w-4\right) \\
\hline
\end{array}$$

4. Compute and simplify:

$$\begin{array}{rl} & -3 \\ - & \left(-5y-5\right) \\ \hline \end{array}$$

5. Compute and simplify:

$$\left(-3w^2-2w-1\right)+\left(3w^4+3w^3-2w^2+6w+3\right)$$

6. Compute and simplify:

$$\left(4r^3+2r^2+4\right)+\left(3r^3-4r^2-2\right)$$

7. Compute and simplify:

$$\begin{array}{rl} & -3r^6+7r^3-5r^2+7r \\ - & \left(-4r^5-5r^4+r\right) \\ \hline \end{array}$$

8. Compute and simplify:

$$\left(2t^3+7t+4\right)+\left(6t^3-2t^2+3t\right)$$

9. Compute and simplify:

$$\begin{array}{rl} & -y^4+y^3+2y^2-4y+5 \\ - & \left(3y^5-y^4+7y^3+7\right) \\ \hline \end{array}$$

10. Compute and simplify:

$$\begin{array}{rl} & 7r^6-5r^5+5r^4-r^2+5r-4 \\ - & \left(r^6+6r^3-5r\right) \\ \hline \end{array}$$

11. Compute and simplify:

$$\begin{array}{rl} & 3r^5+7r^4+5r^2+2 \\ + & -3r^5+3r^4+r^3+r^2+7r+5 \\ \hline \end{array}$$

12. Compute and simplify:

$$(3q-5)+(6q-2)$$

13. Compute and simplify:

$$\begin{array}{rl} & -4v^4-3v^3-3v^2-5v \\ - & \left(-3v^4-v^3+5v\right) \\ \hline \end{array}$$

14. Compute and simplify:

$$\begin{array}{rl} & 4q^6-5q^3+4q-5 \\ - & \left(-2q^6-q^5+7q^3+2\right) \\ \hline \end{array}$$

15. Compute and simplify:

$$(0)+(3t)$$

16. Compute and simplify:

$$\left(-4z^2-5z\right)+\left(-2z^5-z^4-4z^3+5z^2-z+7\right)$$

17. Compute and simplify:

$$\begin{array}{rl} & 5v^2-3 \\ + & 4v^4+6v^3-2v^2-3v+7 \\ \hline \end{array}$$

18. Compute and simplify:

$$\begin{array}{rl} & 2y^2+3y+2 \\ + & 6y^3+6y^2-y \\ \hline \end{array}$$

19. Compute and simplify:

$$\begin{array}{rl} & 1 \\ - & (w+2) \\ \hline \end{array}$$

20. Compute and simplify:

$$\begin{array}{rl} & 7v^2 + 7v \\ + & 6v^2 \\ \hline \end{array}$$

4.2 Multiply Monomials & Binomials

At the heart of multiplication of polynomials is the distributive law. We begin with the easiest case, namely a multiplication of a monomial times a polynomial.

Example Multiply

$$(3x^3)(5x^2 + 4x - 1)$$

solution:

$$\begin{aligned}
&(3x^3)(5x^2 + 4x - 1) \\
&= (3x^3)(5x^2 + 4x + -1) && \text{(def a-b)} \\
&= (3x^3)5x^2 + (3x^3)4x + (3x^3)(-1) && \text{(DL)} \\
&= (3 \cdot 5)x^3x^2 + (3 \cdot 4)x^3x + (3 \cdot -1)x^3 && \text{(ALM, CoLM)} \\
&= 15x^5 + 12x^4 + -3x^3 && \text{(BI)}
\end{aligned}$$

Example Multiply

$$(-2x)(5x^2 + 4x - 1)$$

solution:

$$\begin{aligned}
&(-2x)(5x^2 + 4x - 1) \\
&= (-2x)(5x^2 + 4x + -1) && \text{(def a-b)} \\
&= (-2x)5x^2 + (-2x)4x + (-2x)(-1) && \text{(DL)} \\
&= (-2 \cdot 5)x \cdot x^2 + (-2 \cdot 4)x \cdot x + (-2 \cdot -1)x && \text{(ALM, CoLM)} \\
&= -10x^3 + -8x^2 + 2x && \text{(BI)}
\end{aligned}$$

Example Multiply

$$(-2x + 1)(5x^2 + 4x - 1)$$

solution:

$$
\begin{aligned}
&(-2x+1)(5x^2+4x-1) && \\
&= (-2x+1)(5x^2+4x+-1) && \text{(def a-b)} \\
&= (-2x+1)5x^2+(-2x+1)4x+(-2x+1)(-1) && \text{(DL)} \\
&= (-2x)5x^2+(1)5x^2+(-2x)4x+(1)4x+(-2x)(-1)+(1)(-1) && \text{(DL)} \\
&= -10x^3+5x^2+-8x^2+4x+2x+-1 && \text{(DL)} \\
&= -10x^3+(5x^2+-8x^2)+(4x+2x)+-1 && \text{(ALA)} \\
&= -10x^3+(5+-8)x^2+(4+2)x+-1 && \text{(DL)} \\
&= -10x^3+-3x^2+6x+-1 && \text{(BI)}
\end{aligned}
$$

Alternative solution: Multiplying using the KinderGarden Method:

$$
(-2x+1)(5x^2+4x-1) = \begin{array}{rr}
 & 5x^2+4x+-1 \\
 & \times(-2x+1) \\
\hline
 & 5x^2+4x+-1 \\
+ & -10x^3-8x^2+2x \quad\; \\
\hline
 & -10x^3+-3x^2+6x+-1
\end{array} \qquad \text{(KG M)}
$$

4.2.1 Exercises

1. Compute and simplify using KG:

$$
\begin{array}{rr}
 & u^4+6u^2+3u-5 \\
\times & 7u^2-5u+5 \\
\hline
\end{array}
$$

2. Compute and simplify using KG:

$$
\begin{array}{rr}
 & -5w^5-4w^3+2w \\
\times & 6w^6 \\
\hline
\end{array}
$$

3. Compute and simplify use DL:

$$\left(-2z^7+3z^6+7z\right)\left(-2z^4\right)$$

4. Compute and simplify using KG:

$$\begin{array}{r} y^4+2y^3+7y^2+5y \\ \times \qquad 5y^2+4y-2 \\ \hline \end{array}$$

5. Compute and simplify using KG:

$$\begin{array}{r} w^6-5w^4+7w^3 \\ \times \qquad 7w^6 \\ \hline \end{array}$$

6. Compute and simplify using KG:

$$\begin{array}{r} 4z^3-4z^2-5z+7 \\ \times \qquad 3z^2-2z-4 \\ \hline \end{array}$$

7. Compute and simplify using KG:

$$\begin{array}{r} -w^3+5w^2-3w-4 \\ \times \qquad 4-2w \\ \hline \end{array}$$

8. Compute and simplify using KG:

$$\begin{array}{r} -4v^5+6v^3+3v^2 \\ \times \qquad 3v^5 \\ \hline \end{array}$$

9. Compute and simplify using KG:

$$\begin{array}{r} 3u^4+4u^2-2u-1 \\ \times \qquad u^2-5u-2 \\ \hline \end{array}$$

10. Compute and simplify use DL:

$$\left(t^2+5t-5\right)\left(t^5-2t^2\right)$$

11. Compute and simplify using KG:

$$\begin{array}{r} -4t^3-5t^2-t-2 \\ \times \qquad 6t^2+3t \\ \hline \end{array}$$

12. Compute and simplify using KG:

$$\begin{array}{r} -3r^3+7r^2+1 \\ \times \qquad 7r^3-5r^2 \\ \hline \end{array}$$

13. Compute and simplify using KG:

$$\begin{array}{r} -2z^2+z+4 \\ \times \qquad 2z^3-2z \\ \hline \end{array}$$

14. Compute and simplify using KG:

$$\begin{array}{r} 4r^5+r^3-5r \\ \times \qquad -r \\ \hline \end{array}$$

15. Compute and simplify using KG:

$$\begin{array}{r} 5w^4 + 7w^3 + 6w + 3 \\ \times \qquad - 4w^2 + 4w + 3 \\ \hline \end{array}$$

16. Compute and simplify use DL:

$$\left(-y^5 - 5y^3 - 3y^2\right)(5y)$$

17. Compute and simplify use DL:

$$\left(6z^5 + 4z^3 - 1\right)\left(5z^7\right)$$

18. Compute and simplify using KG:

$$\begin{array}{r} 6z^3 - 4z^2 - 5z \\ \times \qquad 6z^3 - 2z \\ \hline \end{array}$$

19. Compute and simplify use DL:

$$\left(3y^2 + y - 4\right)\left(-3y^3 - 2y\right)$$

20. Compute and simplify using KG:

$$\begin{array}{r} 6y^4 + y + 4 \\ \times \qquad - 4y^2 \\ \hline \end{array}$$

4.3 Multiplying Famous Polynomials

In the previous section we introduced the *kindergarten method* for multiplying polynomials. In this section, we introduce a list of very famous polynomials. In particular, we want to prove, understand, and own every one of these famous products of polynomials. We will see these again, and it will be essential that we recognize them when seen forwards and backwards. Moreover, we introduce a very special kind of multiplication, namely the type when we multiply a binomial times another binomial. For this type of multiplication we introduce, as an option, the FOIL method. As we shall soon see, the FOIL method is simply an abbreviated version of a the distributive law applied a couple times.

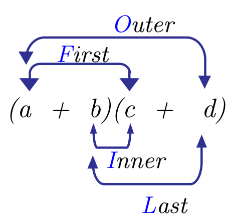
Formally, and without further ado:

Theorem 4.3.1. *FOIL*

Suppose a, b, c, d are numbers or variables for which our axioms apply, then:

$$(a + b)(c + d) = ac + ad + bc + bd \qquad \text{[FOIL]}$$

The name is intended to help one remember the four terms on the right. First we have the first terms from each of the binomials on the left, the a and c are each the First terms thus the F, the the Outter, the Inner, and the Last terms, thus the FOIL acronym.



$$= \overbrace{ac}^{F} + \overbrace{ad}^{O} + \overbrace{bc}^{I} + \overbrace{bd}^{L}$$

Proof.

$$\begin{aligned}(a+b)(c+d) &= a(c+d) + b(c+d) && \text{(DL)}\\ &= a \cdot c + a \cdot d + b \cdot c + b \cdot d && \text{(DL)}\end{aligned}$$

∎

Example

$$\begin{aligned}
(3x+2)(4x+1) &= (3x)(4x)+(3x)(1)+(2)(4x)+(2)(1) && \text{(FOIL)}\\
&= 12x^2+3x+8x+2 && \text{(BI)}\\
&= 12x^2+11x+2 && \text{(BI)}
\end{aligned}$$

One more time...

Example

$$\begin{aligned}
(3x+2)(5x^2+7x) &= (3x)(5x^2)+(3x)(7x)+(2)(5x^2)+(2)(7x) && \text{(FOIL)}\\
&= 15x^3+21x^2+10x^2+14x && \text{(BI)}\\
&= 15x^3+31x^2+14x && \text{(BI)}
\end{aligned}$$

It should be noted that the FOIL method is an option for the case when we are multiplying binomial times a binomial. It becomes less relevant when we are multiplying trinomials or monomials. Meanwhile, the methods from the last section, DL and KG are good for all multiplication of all polynomials from out class.

Next we turn our attention to some very famous polynomial products. We will include here some of the proofs, while some of these and leave the others as important exercises. For all of these, we assume our variables are ones for which our axioms apply.

Theorem 4.3.2. *Difference of Squares [DS]*

$$(a-b)(a+b)=a^2-b^2$$

Theorem 4.3.3. *Difference of two Cubes [DC]*

$$(a-b)(a^2+ab+b^2)=a^3-b^3$$

Proof.

$$\begin{array}{rrrrrl}
(a-b)(a^2+ab+b^2) = & & a^2 & +ab & +b^2 & \text{(KG M)}\\
& & & \times(a & +-b) & \\
\hline
& & -a^2b & +-ab^2 & +-b^3 & \\
+ & a^3 & +a^2b & +ab^2 & & \\
\hline
& a^3+ & 0+ & 0+ & -b^3 & \\
= a^3-b^3 & & & & & \text{(BI)}
\end{array}$$

Theorem 4.3.4. *Difference of two Cubes [SC]*

$$(a+b)(a^2-ab+b^2) = a^3+b^3$$

To prove SC and each of the other polynomials here one may proceed by using DL or KG. While we do a couple of these here, it is most important and beneficial for these modest challenges to be taken on by the reader. Most of these appear as important exercises.

Theorem 4.3.5. *Pascal Polynomials*

- $(a+b)^2 = a^2+2ab+b^2$ *[PP#2]*
- $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ *[PP#3]*
- $(a+b)^4 = a^4+4a^3b+6a^2b^3+4ab^4+b^4$ *[PP#4]*
- $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ *[PP#5]*

Theorem 4.3.6. *Geometric Series Polynomials*

- $(x-1)(x+1) = x^2-1$ *[GS#2]*
- $(x-1)(x^2+x+1) = x^3-1$ *[GS#3]*
- $(x-1)(x^3+x^2+x+1) = x^4-1$ *[GS#4]*
- $(x-1)(x^4+x^3+x^2+x+1) = x^5-1$ *[GS#5]*

As an example we can prove GS#5, and we can prove it at least a couple ways, by KG method or by using DL a couple times. Here is the proof using KG.

Proof.

$$\begin{array}{r}
x^4+x^3+x^2+x+1 \\
\times \qquad x-1 \\
\hline
-x^4-x^3-x^2-x-1 \\
x^5+x^4+x^3+x^2+x \qquad \\
\hline
x^5-1
\end{array}$$

∎

Theorem 4.3.7. *General Geometric Series Polynomials*

- $(x-y)(x+y) = x^2 - y^2$ *[GGS#2]*
- $(x-y)(x^2+xy+y^2) = x^3 - y^2$ *[GGS#3]*
- $(x-y)(x^3+x^2y+xy^2+y^3) = x^4 - y^4$ *[GGS#4]*
- $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) = x^5 - y^5$ *[GGS#5]*

4.3.1 Exercises

1. State and prove the famous polynomial product, [GGS2]

2. Compute and simplify the following product, using FOIL:

$$(7w+5)(5-4w)$$

3. Compute and simplify the following product, using FOIL:

$$(3t+2)(4t-5)$$

4. Use [PP2] to computing and simplify the following:

$$(v+1)^2$$

5. prove [PP3] by computing and simplifying the following:

$$(w+y)^3$$

6. Compute and simplify the following [famous] product:

$$(\beta+1)(\beta-1)$$

7. Compute and simplify the following product, using FOIL:

$$(3q-1)(3-q)$$

8. Use [PP2] to computing and simplify the following:

$$\left(2r^3+3\right)^2$$

9. Compute and simplify the following product, using FOIL:

$$(5z+1)(7z+4)$$

10. Compute and simplify the following [famous] product:

$$\left(5u^2+u\right)\left(5u^2-u\right)$$

11. prove [PP3] by computing and simplifying the following:

$$(r+z)^3$$

12. Use [PP2] to computing and simplify the following:

$$\left(4r^3+7r\right)^2$$

13. Prove [PP2] by computing and simplify the following:

$$(\theta+\phi)^2$$

14. Prove [PP2] by computing and simplify the following:

$$(\beta+\phi)^2$$

15. Compute and simplify the following product, using FOIL:

$$\left(6q^4-q^3\right)\left(7q^6-5q^7\right)$$

16. Use [PP3] to computing and simplify the following:

$$(4-y)^3$$

17. Use [PP3] to computing and simplify the following:

$$(4-z)^3$$

18. Use [PP2] to computing and simplify the following:

$$\left(7v^3+2\right)^2$$

19. Use [PP2] to computing and simplify the following:

$$\left(3-4t^3\right)^2$$

20. State and prove the famous polynomial product, [GS2]

21. Use [PP2] to computing and simplify the following:

$$(3y+1)^2$$

22. Compute and simplify the following product, using FOIL:

$$(2v+5)(4-5v)$$

23. Compute and simplify the following [famous] product:

$$\left(2q^6+4q^2\right)\left(2q^6-4q^2\right)$$

24. Compute and simplify the following [famous] product:

$$\left(7z^5-z^4\right)\left(7z^5+z^4\right)$$

25. Compute and simplify the following [famous] product:

$$\left(11z^5+4z^2\right)\left(11z^5-4z^2\right)$$

26. Compute and simplify the following product, using FOIL:

$$(-4u-3)(-5u-3)$$

27. State and prove the famous polynomial product, [GS2]

28. Compute and simplify the following [famous] product:

$$(11v+5)(11v-5)$$

29. Prove [PP2] by computing and simplify the following:

$$(\beta+\omega)^2$$

30. Compute and simplify the following product, using FOIL:

$$\left(5y^5-y^7\right)\left(7y^6-y^7\right)$$

31. Compute and simplify the following [famous] product:

$$(\mu + 3)(\mu - 3)$$

32. Use [PP2] to computing and simplify the following:

$$\left(6q - 2q^3\right)^2$$

33. Compute and simplify the following [famous] product:

$$(\mu + 1)(\mu - 1)$$

34. Compute and simplify the following product, using FOIL:

$$(3 - 4r)(1 - r)$$

35. Prove [DC] by computing and simplify the following:

$$(\mu + \phi)(\phi - \mu)$$

4.4 Dividing by a Monomial

We will now learn a basic methods used to divide polynomials. The task can be divided into two categories, dividing by a monomial, and dividing my a non-monomial. We will practice dividing by a monomial in this section and leave the dividing by a general polynomial for the next section. We do not need to learn any new axioms or definition, as all the necessary axioms and definition which apply to the real numbers apply to polynomials.

We have harnessed all the necessary ideas already to the point that all left to do is to take a look a couple examples. Once you become good at *looking* at the example you will be ready to practice *doing* some of these yourself in the provided exercises.

Example divide $6x^4 \div 2x$

$$\begin{aligned}
6x^4 \div 2x &= \frac{6x^4}{2x} && \text{(def } \div\text{)} \\
&= \frac{3 \cdot 2x^3x^1}{2x} && \text{(JAE,TT)} \\
&= \frac{3x^3 \cdot 2x}{1 \cdot 2x} && \text{(CoLM, N-Expo, Mid)} \\
&= \frac{3x^3}{1}\frac{2x}{2x} && \text{(MAT)} \\
&= \frac{3x^3}{1} && \text{(JOT,Mid)} \\
&= 3x^3 && \text{(OUT)}
\end{aligned}$$

Notice, this process is nearly identical to the work we did in simplifying rational numbers, where we prime factorized each the numerator & denominator and we eliminated the gcd, ie the common prime pieces. Let's try it one more time.

Example divide $16x^4 \div 2x^3$

$$\begin{aligned}
16x^4 \div 2x^3 &= \frac{16x^4}{2x^3} && \text{(Def } \div\text{)} \\
&= \frac{8 \cdot 2x^1 x^3}{1 \cdot 2x^3} && \text{(JAE,TT)} \\
&= \frac{8x \cdot 2x^3}{1 \cdot 2x^3} && \text{(CoLM, +Expo)} \\
&= \frac{8x}{1} \frac{2x^3}{2x^3} && \text{(MAT)} \\
&= \frac{8x}{1} \cdot 1 && \text{(JOT)} \\
&= 8x && \text{(OUT,Mid)}
\end{aligned}$$

Next, we turn our attention to diving a general *polynomial by a monomial.*

Example divide $(4x^3 + 10x^2 + 8x + 1) \div 2x^2$

$$\begin{aligned}
&(4x^3 + 10x^2 + 8x + 1) \div 2x^2 \\
&= \frac{4x^3 + 10x^2 + 8x + 1}{2x^2} && \text{(def } \div\text{)} \\
&= \frac{4x^3}{2x^2} + \frac{10x^2}{2x^2} + \frac{8x}{2x^2} + \frac{1}{2x^2} && \text{(ATT)} \\
&= 2x + 5 + \frac{4}{x} + \frac{1}{2x^2} && \text{(BI)}
\end{aligned}$$

Example divide $(6x^3 + 4x^2 + 1) \div 2x$

$$
\begin{aligned}
(6x^3+4x^2+1) \div 2x &= \frac{6x^3+4x^2+1}{2x} && \text{(Def a/b, } \div\text{)}\\
&= \frac{6x^3}{2x} + \frac{4x^2}{2x} + \frac{1}{2x} && \text{(ATT)}\\
&= \frac{2x \cdot 3x^2}{2x \cdot 1} + \frac{2x \cdot 2x}{1 \cdot 2x} + \frac{1}{2x} && \text{(JAE,CoLM,BI)}\\
&= \frac{2x}{2x}\frac{3x^2}{1} + \frac{2x}{1}\frac{2x}{2x} + \frac{1}{2x} && \text{(MAT)}\\
&= 1 \cdot \frac{3x^2}{1} + \frac{2x}{1} \cdot 1 + \frac{1}{2x} && \text{(JOT)}\\
&= 3x^2 + 2x + \frac{1}{2x} && \text{(Mid, OUT)}
\end{aligned}
$$

We can conclude $(6x^3+4x^2+1) \div 2x = 3x^2+2x+\frac{1}{2x}$. Alternatively, we say $(6x^3+4x^2+1) \div 2x = 3x^2+2x$ with reminder 1. This method used the idea that *every division problem can be expressed as a fraction,* while making use of our fraction skills. While it has its virtues, it also has drawbacks. This method may not be very useful when we try to divide *polynomial by polynomial.* Ultimately, we will have to resort to *long division,* which is king in the world of $\mathbb{Q}[x]$ under division. Here is a second look at the same problem above, solved using long division.

Example divide $(6x^3+4x^2+1) \div 2x$

First we set it up as, we do when we divide integers using long division,

$$2x\overline{\big)\; 6x^3 + 4x^2 \;} \; +1$$

then the idea is to see howmany times $2x$ goes into the leading term $6x^3$. In other words we calculate $\frac{6x^3}{2x}$. We can do this by inspection [BI] or as example 1, above. In either case, we conclude $\frac{6x^3}{2x} = 3x^2$. This becomes the first part of the quotient.

$$\begin{array}{r} 3x^2 \\ 2x\overline{\big)\; 6x^3 + 4x^2} \; +1 \end{array}$$

It is customary to try to keep columns ordered by degree. Observe the $3x^2$ was placed on the degree 2 column. The next step, as with integers, is to multiply the $3x^2$ by the divisor $2x$ and subtract it from the leading term $6x^3$. The result is shown,

$$\begin{array}{r|l} \multicolumn{1}{r}{} & \qquad\quad 3x^2 \\ \cline{2-2} 2x) & 6x^3 + 4x^2 \quad +1 \\ & -6x^3 \\ \cline{2-2} \end{array}$$

We subtract and bring down the next term $4x^2$ to obtain:

$$\begin{array}{r|l} \multicolumn{1}{r}{} & \qquad\quad 3x^2 \\ \cline{2-2} 2x) & 6x^3 + 4x^2 \quad +1 \\ & -6x^3 \\ \cline{2-2} \multicolumn{1}{r}{} & \qquad\quad 4x^2 \end{array}$$

We now ask how many times will $2x$ go into $4x^2$? That is, we find $\frac{4x^2}{2x} = 2x$. This becomes the second part of the quotient. We obtain

$$\begin{array}{r|l} \multicolumn{1}{r}{} & \qquad\quad 3x^2 + 2x \\ \cline{2-2} 2x) & 6x^3 + 4x^2 \quad +1 \\ & -6x^3 \\ \cline{2-2} \multicolumn{1}{r}{} & \qquad\quad 4x^2 \end{array}$$

The next step is to multiply the $2x$ in the quotient by the divisor, $2x$ to obtain $4x^2$ which we write down below and subtract from $4x^2$ to obtain

$$\begin{array}{r|l} \multicolumn{1}{r}{} & \qquad\quad 3x^2 + 2x \\ \cline{2-2} 2x) & 6x^3 + 4x^2 \quad +1 \\ & -6x^3 \\ \cline{2-2} \multicolumn{1}{r}{} & \qquad\quad 4x^2 \\ \multicolumn{1}{r}{} & \qquad -4x^2 \\ \cline{2-2} \end{array}$$

Finally, we bring down the 1. Since $2x$ does not divide 1 evenly ($\frac{1}{2x}$ does not reduce) it is left as a remainder. We conclude with the expected, $(6x^3 + 4x^2 + 1) \div 2x = 3x^2 + 2x + \frac{1}{2x}$ OR $(6x^3 + 4x^2 + 1) \div 2x = 3x^2 + 2x$ with reminder 1.

A couple more examples are in order

Example divide $(3x^3 + 15x^2 - 6x - 12) \div (3x)$

$$
\begin{array}{rllll}
 & & x^2 & +5x & -2 \\ \cline{2-4}
3x) & 3x^3 & +15x^2 & -6x & -12 \\
 & -3x^3 & & & \\ \cline{2-2}
 & & 15x^2 & & \\
 & & -15x^2 & & \\ \cline{3-3}
 & & & -6x & \\
 & & & 6x & \\ \cline{4-4}
\end{array}
$$

Finally we bring down the -12 and find the remainder is -12, thus $(3x^3 + 15x^2 - 6x - 12) \div (3x) = x^2 + 5x - 2 + \frac{{}^-12}{3x}$

Example $(3x^3 + 5x^2 - 16x - 2) \div (3x)$

$$
\begin{array}{rllll}
 & & x^2 & +\frac{5}{3}x & -\frac{16}{3} \\ \cline{2-4}
3x) & 3x^3 & +5x^2 & -16x & -2 \\
 & -3x^3 & & & \\ \cline{2-2}
 & & 5x^2 & & \\
 & & -5x^2 & & \\ \cline{3-3}
 & & & -16x & \\
 & & & 16x & \\ \cline{4-4}
\end{array}
$$

Finally we bring down the (remainder) -2 and conclude, by *Long Division [LD]* $(3x^3 + 5x^2 - 16x - 2) \div (3x) = x^2 + \frac{5}{3}x + \frac{{}^-16}{3} + \frac{{}^-2}{3x}$

4.4.1 Exercises

1. Divide using Long Division:

$$\left(36y^4 - 48y^3 - 30y^2 - 18y + 27\right) \div (3y)$$

2. Divide using Long Division:

$$\left(-24t^4 + 42t^3 + 30t^2 - 36t + 9\right) \div (-t)$$

3. Divide using [ATT] and [def $\div$]:

$$\left(-2q^9 + 8q^8 - 20q^2\right) \div \left(3q^2\right)$$

4. Divide using Long Division:

$$\left(6u^4 + 51u^3 + 36u^2 + 24u + 54\right) \div (2u)$$

5. Divide using Long Division:

$$\left(51y^4 + 18y^3 + 36y^2 - 48y - 12\right) \div (-y)$$

6. Divide using [ATT] and [def ÷]:

$$\left(23w^8 - 2w^6 + 11w^5\right) \div \left(-2w^3\right)$$

7. Divide using Long Division:

$$\left(3u^4 - 39u^3 + 33u^2 - 30u - 9\right) \div (2u)$$

8. Divide using [ATT] and [def ÷]:

$$\left(-17y^6 - 22y^4 - 12y^3\right) \div \left(-2y^3\right)$$

9. Divide using Long Division:

$$\left(-6y^4 + 48y^3 + 60y^2 + 39y - 33\right) \div (y)$$

10. Divide using [ATT] and [def ÷]:

$$\left(-26v^5 + 11v^4 + 14v^2\right) \div \left(-2v^2\right)$$

11. Divide using Long Division:

$$\left(45y^4 - 12y^3 - 24y^2 + 51y - 36\right) \div (-3y)$$

12. Divide using [ATT] and [def ÷]:

$$\left(-20v^7 + 15v^5 - 18v^3\right) \div \left(-3v^2\right)$$

13. Divide using Long Division:

$$\left(12x^4 - 57x^3 + 21x^2 - 30x + 6\right) \div (-2x)$$

14. Divide using [ATT] and [def ÷]:

$$\left(2z^8 - 6z^7 + 26z^6\right) \div (3z)$$

15. Divide using Long Division:

$$\left(-33x^4 + 24x^3 - 30x^2 - 57x + 42\right) \div (3x)$$

16. Divide using Long Division:

$$\left(-3t^4 + 48t^3 + 27t^2 - 18t + 54\right) \div (-3t)$$

17. Divide using Long Division:

$$\left(-48y^4 - 57y^3 + 12y^2 - 15y - 27\right) \div (3y)$$

18. Divide using [ATT] and [def ÷]:

$$\left(-22r^{10} + 15r^9 - 12r^6\right) \div \left(-2r^3\right)$$

19. Divide using [ATT] and [def ÷]:

$$\left(26t^7 + 13t^5 + 23t^3\right) \div (-3t)$$

20. Divide using Long Division:

$$\left(-15u^4 - 33u^3 + 42u^2 + 36u + 60\right) \div (-u)$$

21. Divide using [ATT] and [def ÷]:

$$\left(-v^9 + 19v^7 - 14v^2\right) \div \left(-v^3\right)$$

22. Divide using Long Division:

$$\left(33u^4 - 27u^3 - 30u^2 - 60u - 54\right) \div (-u)$$

23. Divide using Long Division:

$$\left(-27y^4 - 42y^3 + 18y^2 + 30y + 21\right) \div (2y)$$

24. Divide using [ATT] and [def $\div$]:

$$\left(-11y^8 + 3y^5 - 22y^2\right) \div \left(3y^3\right)$$

25. Divide using Long Division:

$$\left(15x^4 + 21x^3 - 39x^2 - 9x - 3\right) \div (-2x)$$

4.5 Dividing by Polynomials ii

Last section we divided polynomials by *monomials* exclusively. We now tackle the general problem of dividing by a *binomial* or any polynomial. Consider dividing using long division $2 \div 5$. If you think about it for a second you will realize the quotient is 0 with a remainder of 2. This will usually happen when trying to divide an small integer by a larger one. The same holds for polynomials. You will not get very far if you are trying to divide polynomial of small degree by a polynomial of large degree. The quotient will be 0 with remainder equal to the remainder being the original polynomial. For example, $(5x + 1) \div x^3 = 0$ with remainder $5x + 1$ OR $\frac{5x+1}{x^3}$. Therefore, we will practice our diving skill mostly with cases where we divide a polynomial of large degree by one of smaller degree. The Long Division method will be the primary tool.

Example Divide

$$(3x^3 + 5x^2 - 6x - 2) \div (x - 1)$$

Most steps are identical. One small difference is that we always concentrate on the leading terms, momentarily ignoring the other term. For example, we just concentrate on the leading term $3x^3$ inside and x, the leading term inside. We calculate by inspection $\frac{3x^3}{x} = 3x^2$ to obtain our first term of the quotient.

$$\begin{array}{r|l} & \phantom{3x^3 + {}}3x^2 \\ \hline x-1) & 3x^3 + 5x^2 - 6x - 2 \end{array}$$

Now, we multiply the $3x^2$ by the entire divisor $(x - 1)$, thus we have to use DL, to get $3x^2(x-1) = 3x^3 - 3x^2$. This is the quantity we subtract (i.e. change the sign). We obtain...

$$\begin{array}{r|l} & \phantom{3x^3 + {}}3x^2 \\ \hline x-1) & 3x^3 + 5x^2 - 6x - 2 \\ & \underline{-3x^3 + 3x^2} \end{array}$$

Since we have changed the signs already (to subtract), we can simply add the terms and bring down the next term to obtain,

$$\begin{array}{r|l} & \phantom{3x^3 + {}}3x^2 \\ \hline x-1) & 3x^3 + 5x^2 - 6x - 2 \\ & \underline{-3x^3 + 3x^2} \\ & \phantom{3x^3 + {}}8x^2 - 6x \end{array}$$

We now calculate the quotient for leading terms $\frac{8x^2}{x} = 8x$ to get the next term of the quotient...

$$\begin{array}{r}
3x^2 + 8x \phantom{{}-2} \\
x-1 \overline{\smash{\big)}\, 3x^3 + 5x^2 - 6x - 2} \\
\underline{-3x^3 + 3x^2} \phantom{{}-6x-2} \\
8x^2 - 6x \phantom{{}-2}
\end{array}$$

We now multiply $8x(x-1) = 8x^2 - 8x$ and subtract, OR change sign and add to obtain....

$$\begin{array}{r}
3x^2 + 8x \phantom{{}-2} \\
x-1 \overline{\smash{\big)}\, 3x^3 + 5x^2 - 6x - 2} \\
\underline{-3x^3 + 3x^2} \phantom{{}-6x-2} \\
8x^2 - 6x \phantom{{}-2} \\
\underline{-8x^2 + 8x} \phantom{{}-2} \\
2x - 2
\end{array}$$

Finally, we calculate $\frac{2x}{x} = 2$ for the last part of the quotient...

$$\begin{array}{r}
3x^2 + 8x + 2 \\
x-1 \overline{\smash{\big)}\, 3x^3 + 5x^2 - 6x - 2} \\
\underline{-3x^3 + 3x^2} \phantom{{}-6x-2} \\
8x^2 - 6x \phantom{{}-2} \\
\underline{-8x^2 + 8x} \phantom{{}-2} \\
2x - 2
\end{array}$$

Then we have

$$\begin{array}{r}
3x^2 + 8x + 2 \\
x-1 \overline{\smash{\big)}\, 3x^3 + 5x^2 - 6x - 2} \\
\underline{-3x^3 + 3x^2} \phantom{{}-6x-2} \\
8x^2 - 6x \phantom{{}-2} \\
\underline{-8x^2 + 8x} \phantom{{}-2} \\
2x - 2 \\
\underline{-2x + 2} \\
0
\end{array}$$

Therefore, by [LD] we have $(3x^3 + 5x^2 - 6x - 2) \div (x-1) = 3x^2 + 8x + 2$

Example Divide $(10x^4 + 13x^3 + 8x^2 + 8x + 3) \div (2x + 1)$

$$
\begin{array}{rl}
 & 5x^3 + 4x^2 + 2x + 3 \\ \cline{2-2}
2x+1 \big) & 10x^4 + 13x^3 + 8x^2 + 8x + 3 \\
 & -\,10x^4 \;-\; 5x^3 \\ \cline{2-2}
 & 8x^3 + 8x^2 \\
 & -\,8x^3 - 4x^2 \\ \cline{2-2}
 & 4x^2 + 8x \\
 & -\,4x^2 - 2x \\ \cline{2-2}
 & 6x + 3 \\
 & -\,6x - 3 \\ \cline{2-2}
 & 0
\end{array}
$$

Example

Divide $(4x^3 +^- 4x^2 +^- 23x + 22) \div (2x - 5)$

$$
\begin{array}{rl}
 & 2x^2 \;+ 3x \;- 4 \\ \cline{2-2}
2x-5 \big) & 4x^3 \;- 4x^2 - 23x + 22 \\
 & -\,4x^3 + 10x^2 \\ \cline{2-2}
 & 6x^2 - 23x \\
 & -\,6x^2 + 15x \\ \cline{2-2}
 & -\,8x + 22 \\
 & 8x - 20 \\ \cline{2-2}
 & 2
\end{array}
$$

thus by [LD] $(4x^3 +^- 4x^2 +^- 23x + 22) \div (2x - 5) = 2x^2 + 3x - 4 + \frac{2}{2x-5}$

Example Divide $(4x^4 + 8x^3 + -3x^2 - x - 4) \div (2x^2 + 3x - 4)$

$$
\begin{array}{rl}
 & 2x^2 + 3x - 4 \\ \cline{2-2}
2x^2+x+1 \big) & 4x^4 + 8x^3 - 3x^2 \;- x - 4 \\
 & -\,4x^4 - 2x^3 - 2x^2 \\ \cline{2-2}
 & 6x^3 - 5x^2 \;- x \\
 & -\,6x^3 - 3x^2 - 3x \\ \cline{2-2}
 & -\,8x^2 - 4x - 4 \\
 & 8x^2 + 4x + 4 \\ \cline{2-2}
 & 0
\end{array}
$$

thus by [LD] $(4x^3 +^- 4x^2 +^- 23x + 22) \div (2x - 5) = 2x^2 + 3x - 4$

Example Divide $(4x^4 + 8x^3 + -3x^2 - x - 4) \div (3x + 2)$

$$
\begin{array}{rrrrrrr}
 & & \frac{4}{3}x^3 & +\frac{16}{9}x^2 & -\frac{59}{27}x & +\frac{91}{81} \\
\cline{2-6}
3x+2 \big) & 4x^4 & +8x^3 & -3x^2 & -x & -4 \\
 & -4x^4 & -\frac{8}{3}x^3 & & & \\
\cline{2-3}
 & & \frac{16}{3}x^3 & -3x^2 & & \\
 & & -\frac{16}{3}x^3 & -\frac{32}{9}x^2 & & \\
\cline{3-4}
 & & & -\frac{59}{9}x^2 & -x & \\
 & & & \frac{59}{9}x^2 & +\frac{118}{27}x & \\
\cline{4-5}
 & & & & \frac{91}{27}x & -4 \\
 & & & & -\frac{91}{27}x & -\frac{182}{81} \\
\cline{5-6}
 & & & & & -\frac{506}{81}
\end{array}
$$

4.5.1 Exercises

1. Divide using Long Division:

$$\left(-96t^2+40t-1\right) \div (8t-2)$$

2. Divide using Long Division:

$$\left(-84x^6+68x^5+92x^4-59x^3+26x^2-x-50\right) \div \left(-7x^2+x+6\right)$$

3. Divide using Long Division:

$$\left(30y^6+76y^5+81y^4+40y^3-20y^2+19y-9\right) \div \left(-6y^2-8y+3\right)$$

4. Divide using Long Division:

$$\left(90y^3+48y^2+33y+12\right) \div (-9y-3)$$

5. Divide using Long Division:

$$\left(-12y^4+70y^3-88y^2-42y+71\right) \div (6y-11)$$

6. Divide using Long Division:

$$\left(-20x^5-105x^3-95x^2+60x+31\right) \div (10x+5)$$

7. Divide using Long Division:

$$\left(132u^5 + 88u^4 + 12u^3 - 16u^2 + 128u + 98\right) \div (-12u - 8)$$

8. Divide using Long Division:

$$\left(40y^4 - 72y^3 - 78y^2 + 36y - 34\right) \div (4y - 10)$$

9. Divide using Long Division:

$$\left(-9u^2 + 6u + 23\right) \div (-3u - 4)$$

10. Divide using Long Division:

$$\left(5y^5 - 10y^4 + 6y^3 - 24y^2 + 15y + 25\right) \div (2 - y)$$

11. Divide using Long Division:

$$\left(99y^5 + 76y^4 + 84y^3 - 11y^2 - 87y - 25\right) \div (-9y - 2)$$

12. Divide using Long Division:

$$\left(-28y^5 + 63y^4 - 20y^3 + 13y^2 + 56y + 28\right) \div (9 - 4y)$$

13. Divide using Long Division:

$$\left(-24u^4 - 36u^3 - 48u^2 - 96u + 88\right) \div (-6u - 12)$$

14. Divide using Long Division:

$$\left(99y^3 - 63y^2 - 33y + 17\right) \div \left(3 - 9y^2\right)$$

15. Divide using Long Division:

$$\left(-55u^5 + 96u^4 + 90u^3 - 62u^2 - 53u + 45\right) \div (5u - 11)$$

4.6 Factor Polynomials by DL

Recall 'To factor' means to break up into multiples. The main tool here will be the distributive law. To *factor completely* means to factor, factor until you can factor no more. Another name for factoring completely is prime factorization. Here we may not learn to all methods to factor polynomials *completely.* However, we will learn *one* method of factoring polynomials. This method called the *DL METHOD* involves looking at each of the terms, finding the gcd for the collection of terms, and factoring this gcd using DL among other axioms and theorems. If the gcd is 1, then this method does not provide any non-trivial factorization of the polynomial.

Example factor $6x + 4$

note the $\gcd(6x, 4) = 2$, thus using the DL method we can factor 2 from each of the terms.

$$\begin{aligned} 6x + 4 &= 2 \cdot 3x + 2 \cdot 2 && \text{(T.T)} \\ &= 2(3x + 2) && \text{(D.L.)} \end{aligned}$$

Example factor $16x^2 + 40x + 24$.

note the $\gcd(16x^2, 40x, 24) = 8$, thus using the DL method we can factor 8 from each of the terms.

$$\begin{aligned} 16x^2 + 40x + 24 &= 8 \cdot 2x^2 + 8 \cdot 5x + 8 \cdot 3 && \text{(T.T)} \\ &= 8(2x^2 + 5x + 3) && \text{(D.L.)} \end{aligned}$$

Example factor $16x^5 + 40x^4 + 24x^3$

note the $\gcd(16x^5, 40x^4, 24x^3) = 8x^3$, thus using the DL method we can factor $8x^3$ from each of the terms.

$$\begin{aligned} 16x^5 + 40x^4 + 24x^3 &= 8x^3 \cdot 2x^2 + 8x^3 \cdot 5x + 8x^3 \cdot 3 && \text{(B.I.)} \\ &= 8x^3(2x^2 + 5x + 3) && \text{(D.L.)} \end{aligned}$$

Example factor $(blah)x^3 + (blah)2x$

$$\begin{aligned} (blah)x^3 + (blah)2x &= (blah)(x^3 + 2x) && \text{(D.L.)} \\ &= (blah)(x^2 + 2)x && \text{(D.L., BI)} \end{aligned}$$

Example factor $(x+5)x^3 + (x+5)2x$

$$
\begin{aligned}
&(x+5)x^3 + (x+5)2x && \text{(given)}\\
&= (x+5)(x^3 + 2x) && \text{(D.L.)}\\
&= (x+5)(x^2 \cdot x^2 + 2x) && \text{(J.A.E.)}\\
&= (x+5)(x^2 \cdot x + 2x) && \text{(N.Expo)}\\
&= (x+5)(x^2 + 2)x && \text{(D.L., BI)}
\end{aligned}
$$

It should be noted that this method for factoring is not always successful. Specially if there is nothing common amongst the terms.

Example factor $(x+4)x^3 + (x+5)2x$

$$
\begin{aligned}
&(x+4)x^3 + (x+5)2 && \text{(given)}\\
=&(x+4)x^3 + (x+5)2 && \text{(nothing we can factor)}
\end{aligned}
$$

This expression is as factored as it will get, at least using the DL method from this section.

As always, the homework is an essential part of the learning process. At this point the only thing left to do is to get to it. The exercises should help you practice spotting *gcd*'s and factoring it from each of the terms using DL.

4.6.1 Exercises

1. Factor using DL:

$$8t^{11} - 11t^6 - 7t^3$$

2. Factor using DL:

$$3u^5 - 5u^4$$

3. Factor using DL:

$$9t^{12} + 7t^{7} + 8t^{4}$$

4. Factor using DL:

$$-8t^{6} + 11t^{3} + 10t$$

5. Factor using DL:

$$12y^{7} - 7y^{4} - 8y^{2}$$

6. Factor using DL:

$$-5\left(t^{3}\right) - 9\left(t^{3}\right)u$$

7. Factor using DL:

$$11u^{5}\left(u^{3}\right) - 12u^{2}\left(u^{3}\right) + 7\left(u^{3}\right)$$

8. Factor using DL:

$$9t^{8} + 5t^{5} - 4t^{3}$$

9. Factor using DL:

$$-6y^{12} + 4y^{7} + 7y^{4}$$

10. Factor using DL:

$$12t^{5} - 2t^{2} + 7t$$

11. Factor using DL:

$$12\left(x^{3}+8\right) + 8\left(x^{3}+8\right)u^{5} - \left(x^{3}+8\right)u^{2}$$

12. Factor using DL:

$$t^{12} + 12t^{7} + 8t^{4}$$

13. Factor using DL:

$$3\left(5x+12\right)u - 8\left(5x+12\right)$$

14. Factor using DL:

$$7t^8 + 3t^5 - 8t^3$$

15. Factor using DL:

$$5u^2\omega^2 + u\omega^2 - 8\omega^2$$

16. Factor using DL:

$$y^4 - 12y^5$$

17. Factor using DL:

$$7t^2 - 9t$$

18. Factor using DL:

$$10\mu^4 + 8\mu^4x^3 - 12\mu^4x^2 + 3\mu^4x$$

19. Factor using DL:

$$11t^8\,(11u+2) - 7t^3\,(11u+2) - (11u+2)$$

20. Factor using DL:

$$7t^5\left(x^3\right) - 2t^2\left(x^3\right) - 11\left(x^3\right)$$

21. Factor using DL:

$$2\left(x^3\right) - 4\left(x^3\right)t^4 + \left(x^3\right)t$$

22. Factor using DL:

$$-7\lambda^4 - 8\lambda^4x$$

23. Factor using DL:

$$-11\theta^2 - 2\theta^2x$$

24. Factor using DL:

$$-7\,(7y+11) + 4\,(7y+11)\,t^8 + 6\,(7y+11)\,t^3$$

25. Factor using DL:

$$-4\mu^2 - 6\mu^2 t^2 - 3\mu^2 t$$

26. Factor using DL:

$$11x^2 + 4x$$

27. Factor using DL:

$$-10u^7 + 9u^4 + 11u^2$$

28. Factor using DL:

$$11u^4 + 6u^3$$

29. Factor using DL:

$$6u^4 \left(x^6\right) - 9u \left(x^6\right) - 10 \left(x^6\right)$$

30. Factor using DL:

$$12 \left(y^2\right) + 11 \left(y^2\right) x^5 + 9 \left(y^2\right) x^2$$

4.7 Factor By Grouping

In the our section, we learned to factor using DL, once we identified the gcd for the terms of a polynomial. It turns out this method often fails. Particularly, if there is nothing in common within the terms, that is, it fails if the gcd is 1. Such as:

$$3 + 5y + 6y^3 + 10y^4$$

In this section, we discuss one possible remedy for this, It is called *factor by grouping*. While it works with many polynomials, we will pay special attention to the case when we have a quad-nominal, that is a polynomial with four terms such as the one state above. The basic idea is to group the terms, and see where that leads.

Example

$$\begin{aligned} & 3 + 5y + 6y^3 + 10y^4 && \text{(given)} \\ =& (3 + 5y) + \left(6y^3 + 10y^4\right) && \text{(ALA)} \end{aligned}$$

We have grouped but nothing has happened... but wait..

$$\begin{aligned} & 3 + 5y + 6y^3 + 10y^4 && \text{(given)} \\ =& (3 + 5y) + \left(6y^3 + 10y^4\right) && \text{(ALA)} \\ =& 1 \cdot (3 + 5y) + 2y^3 (3 + 5y) && \text{(BI, DL)} \end{aligned}$$

But now we see, both of these groups have something in common, indeed, we proceed,

$$\begin{aligned} & 3 + 5y + 6y^3 + 10y^4 && \text{(given)} \\ =& (3 + 5y) + \left(6y^3 + 10y^4\right) && \text{(ALA)} \\ =& 1 \cdot (3 + 5y) + 2y^3 (3 + 5y) && \text{(BI, DL)} \\ =& \left(1 + 2y^3\right) (3 + 5y) && \text{(DL)} \end{aligned}$$

... and the *factor by grouping method* has succeeded! It does not always work, and it is not always clear when it will work. You must try it and practice. If you can't factor a polynomial using this method, relax. Later we will learn a few more ideas to factor.

Example factor $1+3x+x^2+3x^3$

$$\begin{aligned}
&1+3x+x^2+3x^3 \\
&=(1+3x)+(1\cdot x^2+3x\cdot x^2) && \text{(ALA and M.Id)} \\
&=(1+3x)\cdot 1+(1+3x)x^2 && \text{(MId, D.L.)} \\
&=(1+3x)(1+x^2) && \text{(D.L.)}
\end{aligned}$$

Example factor $1+3x-x^2-3x^3$

$$\begin{aligned}
&1+3x-x^2-3x^3 \\
&=(1+3x)+(-x^2+-3x^3) && \text{(ALA)} \\
&=(1+3x)+(1\cdot -x^2+3x\cdot -x^2) && \text{(MId, N-Expo, CLM)} \\
&=(1+3x)+(1+3x)(-x^2) && \text{(D.L.)} \\
&=(1+3x)\cdot 1+(1+3x)(-x^2) && \text{(M.Id)} \\
&=(1+3x)(1+(-x^2)) && \text{(D.L.)} \\
&=(1+3x)(1+-x^2) && \text{(D.L.(to be continued...))}
\end{aligned}$$

At last, the most important thing to do is for you to practice the *factor by grouping method.*

4.7.1 Exercises

1. Factor by grouping:

$$2+-4x^2+-4x^3+8x^5$$

2. Factor by grouping:

$$12+4\rho^2+3\rho^3+\rho^5$$

3. Factor by grouping:

$$1+-t^3+-t^2+t^5$$

4. Factor by grouping:

$$4 + 3Q^2 + 4Q^3 + 3Q^5$$

5. Factor by grouping:

$$20 + 8w + 5w^2 + 2w^3$$

6. Factor by grouping:

$$2 + 10x + 2x^2 + 10x^3$$

7. Factor by grouping:

$$6 + 10v^2 + 6v^3 + 10v^5$$

8. Factor by grouping:

$$6 + 3v + 10v^2 + 5v^3$$

9. Factor by grouping:

$$16 + 12\theta^2 + 16\theta^3 + 12\theta^5$$

10. Factor by grouping:

$$-2 + 5T^2 + -8T^3 + 20T^5$$

11. Factor by grouping:

$$10 + 20z + 10z^2 + 20z^3$$

12. Factor by grouping:

$$15 + 3T^2 + 15T^3 + 3T^5$$

13. Factor by grouping:

$$3 + 12x^2 + 4x^3 + 16x^5$$

14. Factor by grouping:

$$4 + -10\beta^3 + -6\beta^2 + 15\beta^5$$

15. Factor by grouping:

$$3 + 15\lambda^3 + 5\lambda^2 + 25\lambda^5$$

16. Factor by grouping:

$$12 + 12\xi^3 + 6\xi^2 + 6\xi^5$$

17. Factor by grouping:

$$20 + 15q + 16q^2 + 12q^3$$

18. Factor by grouping:

$$12 + 4\phi^2 + 12\phi^3 + 4\phi^5$$

19. Factor by grouping:

$$-1 + 2\lambda^2 + -4\lambda^3 + 8\lambda^5$$

20. Factor by grouping:

$$-1 + -2U^3 + 4U^2 + 8U^5$$

4.8 Factor by Splitting

In this section, we consider trinomials such as

$$x^2 + 5x + 4$$

which can not be factored by DL, nor by grouping. For these we introduce some brand new and very creative medicine. We call this medicine *split the middle* term. Here is roughly what the process looks like in action.

$$\begin{aligned} & x^2 + 5x + 4 && \text{(given)} \\ =& x^2 + (1+4)x + 4 && \text{(AT)} \\ =& x^2 + 1x + 4x + 4 && \text{(DL)} \end{aligned}$$

We have indeed split the middle term. You may wonder, while all steps look legal, what madness would drive one to do such a silly thing as splitting the middle term? As it turns out this becomes the key, turning the trinomial into a quadnomial on which we can factor by grouping, much like we did on the last section. Observe:

$$\begin{aligned} & x^2 + 5x + 4 && \text{(given)} \\ =& x^2 + (1+4)x + 4 && \text{(AT)} \\ =& x^2 + 1x + 4x + 4 && \text{(DL)} \\ =& \left(x^2 + 1x\right) + (4x + 4) && \text{(ALA)} \\ =& (x+1)\,x + (x+1)\,4 && \text{(BI, DL)} \\ =& (x+1)\,(x+4) && \text{(BI, DL)} \end{aligned}$$

viola! We factored by splitting the middle term. It may help to re-read this example, and try to pinpoint the key to its success. The key is arguably the splitting of $5x$ into $1x + 4x$. This may lead you to wonder, how did we know that would work? would other splitting also work? where did this come from? For starters, not all splitting works to end up with a nicely factored polynomial, for example had we tried to split $5x$ into $3x + 2x$, though legal, we would have gotten stuck on this step:

$$
\begin{aligned}
&x^2 + 5x + 4 && \text{(given)}\\
=&x^2 + (1+4)x + 4 && \text{(AT)}\\
=&x^2 + 1x + 4x + 4 && \text{(DL)}\\
=&\left(x^2 + 3x\right) + (2x + 4) && \text{(ALA)}\\
=&(x+3)\,x + (x+2)\,2 && \text{(ALA)}
\end{aligned}
$$

Because the terms do not match, we can not complete the factorization by grouping strategy. This shows that not all splitting works, which begs the question, which do?

Here is one way to see which splitting work. Consider working backwards. That is suppose we had already factored, using some numbers we yet don't know but imagine, a, b, c, d it might look like this:

$$x^2 + 5x + 4 = (ax + b)(cx + d)$$

Then we try to gather hints about these possible numbers, a, b, c, d.

$$
\begin{aligned}
x^2 + 5x + 4 &= (ax+b)(cx+d) && \text{(Assume)}\\
&= acx^2 + bcx + dax + bd && \text{(FOIL)}
\end{aligned}
$$

Now comes the hint, if you multiply $bc \cdot da$ you get the same thing as if you multiply $ac \cdot bd$ Said differently, the split coefficients, when multiplied give you the same as the outer coefficients when multiplied, when multiplied. Said differently the split 5 pieces must be the same as 4, the product of outer coefficients. So to look for the possible split pieces of 5 we look to the possible factors of 4. Namely,

$$
\begin{aligned}
4 &= 2 \cdot 2 && \text{(or)}\\
4 &= -2 \cdot -2 && \text{(or)}\\
4 &= 1 \cdot 4 && \text{(or)}\\
4 &= -1 \cdot -4
\end{aligned}
$$

We see the product $1 \cdot 4 = 4$ and the sum $1 + 4 = 5$, thus we have *winner winner chicken dinner.*

That was lots of explaining, and maybe good to come back and re-read after a couple example. Lets try one more,

Example Factor $3x^2 + 2x - 16$

First consider the outer product $3 \cdot -16 = -48$ then consider the ways of factoring it

$$\begin{aligned} -48 &= 1 \cdot -48 && \text{(or)} \\ -48 &= 2 \cdot -24 && \text{(or)} \\ -48 &= 3 \cdot -16 && \text{(or)} \\ &\ldots\ldots \end{aligned}$$

we're looking for a pair,w/ product "-48" & w/sum "2", so we can split the $2x$

$$\begin{aligned} &\ldots\ldots \\ -48 &= 4 \cdot -12 && \text{(or)} \\ -48 &= -6 \cdot 8 && \text{(winner winner!)} \end{aligned}$$

That was the harder part, now, we do the splitting and all will be good..

$$\begin{aligned} &3x^2 + 2x - 16 && \text{(given)} \\ =&3x^2 + (-6 + 8)x + -16 && \text{(AT)} \\ =&3x^2 + -6x + 8x + -16 && \text{(DL)} \\ =&\left(3x^2 + -6x\right) + (8x + -16) && \text{(ALA)} \\ =&(x + -2)\,3x + (x + -2)\,8 && \text{(DL, BI)} \\ =&(x + -2)\,(3x + 8) && \text{(DL)} \end{aligned}$$

4.8.1 Exercises

1. Factor by Splitting The Middle Term:

$$4\lambda^2 + 20\lambda + 24$$

2. Factor by Splitting The Middle Term:

$$6x^4 + 42x^2 + 60$$

3. Factor by Splitting The Middle Term:

$$z^2 + -4z + 4$$

4. Factor by Splitting The Middle Term:

$$6\beta^2 + 26\beta + 8$$

5. Factor by Splitting The Middle Term:

$$6\mu^6 + 30\mu^3 z^2 + 36z^4$$

6. Factor by Splitting The Middle Term:

$$2\rho^6 + 16\rho^3 r^2 + 30r^4$$

7. Factor by Splitting The Middle Term:

$$2q^2 + 27q + 55$$

8. Factor by Splitting The Middle Term:

$$2x^4 + 23x^2 + 45$$

9. Factor by Splitting The Middle Term:

$$3\gamma^6 + 21\gamma^3 q^2 + 30q^4$$

10. Factor by Splitting The Middle Term:

$$t^2 + 19t + 84$$

11. Factor by Splitting The Middle Term:

$$3\mu^6 + 10\mu^3 t^2 + 3t^4$$

12. Factor by Splitting The Middle Term:

$$6x^2 + -6x + -120$$

13. Factor by Splitting The Middle Term:

$$2\lambda^2 + 17\lambda + 21$$

14. Factor by Splitting The Middle Term:

$$2x^2 + 33x + 121$$

15. Factor by Splitting The Middle Term:

$$-w^2 + 11w + 12$$

16. Factor by Splitting The Middle Term:

$$-2v^2 + 5v + 3$$

17. Factor by Splitting The Middle Term:

$$4q^2 + 12q + 8$$

18. Factor by Splitting The Middle Term:

$$2w^2 + 22w + 60$$

19. Factor by Splitting The Middle Term:

$$6r^4 + 31r^2 + 35$$

20. Factor by Splitting The Middle Term:

$$3\omega^6 + 23t^2\omega^3 + 40t^4$$

21. Factor by Splitting The Middle Term:

$$2\beta^2 + 19\beta + 42$$

22. Factor by Splitting The Middle Term:

$$3\lambda^2 + 19\lambda + 6$$

23. Factor by Splitting The Middle Term:

$$2t^4 + 25t^2 + 72$$

24. Factor by Splitting The Middle Term:

$$2z^2 + -2z + -12$$

25. Factor by Splitting The Middle Term:

$$3\gamma^2 + 4\gamma + 1$$

26. Factor by Splitting The Middle Term:

$$6\theta^6 + 33\theta^3 q^2 + 45q^4$$

27. Factor by Splitting The Middle Term:

$$4x^2 + -30x + 44$$

28. Factor by Splitting The Middle Term:

$$-r^2 + -9r + -18$$

29. Factor by Splitting The Middle Term:

$$3\mu^2 + 5\mu + 2$$

30. Factor by Splitting The Middle Term:

$$-2v^2 + 15v + -27$$

4.9 Factor Famous Polynomials

In this section, we simply pause to reflect on some very famous polynomials. Recall earlier in this chapter we studying the multiplication of famous polynomials. Such as the Pascal Polynomial #2 [PP2] which summarized that

$$(x+y)^2 = x^2 + 2yx + y^2$$

Along with that one we studied many other famous polynomials and we verified these by doing KG multiplication on them, or by using DL to distribute and multiply. In this section, we review the exact same polynomials, but rather than seeing them as famous multiplications, we look at them backwards as famous factorizations. In other words, instead of seeing PP2 as

$$(x+y)(x+y) = x^2 + 2yx + y^2$$

We see it the other way around as a famous factorization:

$$x^2 + 2yx + y^2 = (x+y)(x+y)$$

Since it is really the same statement with a little symmetric property **??** added, we do not need to prove it. We simply want to practice spotting these and become very good and fluent at factoring them. Not only do we want to review and spot all PP2 polynomial forms, but we also want to do the same for the other dozen or so famous polynomials. Here is the list of famous polynomials from earlier this chapter.

Example Suppose we wanted to factor $x^2 + 6x + 9$

While we could split the middle as done in the previous section, we want to become good at spotting patterns. The first term is a square 'x^2', the last terms seems to be a square too, $9 = 3^2$ and the middle has the right form, 2 times each of these, thus it has the perfect PP2 pattern. We proceed as follows:

$$\begin{aligned} (\square + \triangle)^2 &= \square^2 + 2\square\triangle + \triangle^2 && \text{(..recall the pp2 Pattern)} \\ x^2 + 6x + 9 &= x^2 + 2 \cdot 3 \cdot x + 3^2 && \text{(BI (spotted pattern))} \\ &= (x+3)^2 && \text{(PP2)} \end{aligned}$$

.. and we are done! We factored this famous polynomial. We want to practice this sort of factoring with all of our famous polynomials. Here is a list for review. Followed by a couple more examples then the very important exercises.

Very Famous Polynomials

name	Short	Example
Difference of Squares	$[DS]$	$x^2 - y^2 = (x-y)(x+y)$
Difference of Cubes	$[DS]$	$x^3 - y^3 = (x-y)(x^2+xy+y^2)$
Sum of Cubes	$[SC]$	$x^3 + y^3 = (x+y)(x^2-xy+y^2)$
Sum of Cubes	$[PP2]$	$(x+y)^2 = x^2+2xy+y^2$

Pascal Polynomials

name	Short	Example
Pascal Polynomial #2	$[PP2]$	$(x+y)^2 = x^2+2yx+y^2$
Pascal Polynomial #3	$[PP3]$	$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$
Pascal Polynomial #4	$[PP4]$	$(x+y)^4 = x^4+4x^3y+6x^2y^2+6xy^3+y^4$
Pascal Polynomial #5	$[PP5]$	$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

Geometric Series Polynomials

name	Short	Example
Geometric Series #2	$[GS2]$	$x^2-1 = (x-1)(x+1)$
Geometric Series #3	$[GS3]$	$x^3-1 = (x-1)(x^2+x+1)$
Geometric Series #4	$[GS4]$	$x^4-1 = (x-1)(x^3+x^2+x+1)$
Geometric Series #5	$[GS5]$	$x^5-1 = (x-1)(x^4+x^3+x^2+x+1)$

Generalized Geometric Series Polynomials

name	Short	Example
Geometric Series #2	$[GGS2]$	$x^2-y^2 = (x-y)(x+y)$
Geometric Series #3	$[GGS3]$	$x^3-y^3 = (x-y)(x^2+xy+y^2)$
Geometric Series #4	$[GGS4]$	$x^4-y^4 = (x-1)(x^3+x^2y+xy^2+y^3)$
Geometric Series #5	$[GGS5]$	$x^5-y^4 = (x-1)(x^4+x^3y+x^2y^2+xy^3+y^4)$

Example Suppose we wanted to factor $4x^2 - 10x + 25$
We note the first term is a square, $4x^2 = (2x)^2$, the last terms seems to be a square too, $25 = (-5)^2$ and the middle has the right form, 2 times each of these, thus it has the perfect PP2 pattern. We proceed as follows:

$$\begin{aligned}
(\square + \triangle)^2 &= \square^2 + 2\square\triangle + \triangle^2 && \text{(..recall the pp2 Pattern)}\\
4x^2 - 10x + 25 &= (2x)^2 + 2 \cdot (-5) \cdot x + (-5)^2 && \text{(BI (identified PP2 pattern))}\\
&= (2x + -5)^2 && \text{(PP2)}
\end{aligned}$$

.. and we are done! We factored this famous polynomial.

Example Suppose we wanted to factor $8x^2 + \frac{1}{125}$
We note the first term is a cube $8x^3 = (2x)^3$, the last terms seems to be a cube too, $\frac{1}{125} = \left(\frac{1}{5}\right)^3$, thus it has the perfect Sum of Cubes pattern. We proceed as follows:

$$\begin{aligned}
\square^3 + \triangle^3 &= (\square + \triangle)\left(\square^2 - \square\triangle + \triangle^2\right) && \text{(..recall the SC Pattern)}\\
8x^3 + \frac{1}{125} &= (2x)^3 + \left(\frac{1}{5}\right)^3 && \text{(BI (identified SC pattern))}\\
&= \left(2x + \frac{1}{5}\right)\left[(2x)^2 - (2x)\left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)^2\right] && \text{(SC)}
\end{aligned}$$

.. and we are done! We factored another famous polynomial.

Example Suppose we wanted to factor $32x^4 - 243$
We note the first term is a fifth power.. $32x^5 = (2x)^5$, the last terms seems to be a fifth power too, $243 = 3^5$, thus it has the perfect General Geometric #5 pattern. We proceed as follows:

$$\begin{aligned}
\square^5 - \triangle^5 &= (\square - \triangle)\left(\square^4 + \square^3\triangle + \square^2\triangle^2 + \square\triangle^3 + \triangle^4\right)\\
& && \text{(..recall the GGS5 Pattern)}\\
32x^5 - 243 &= (2x)^5 - 3^5 && \text{(BI (identified GG5 pattern))}\\
&= (2x - 3)\left[(2x)^4 + (2x)^3(3) + (2x)^2(3)^2 + (2x)(3)^3 + (3)^4\right] && \text{(SC)}
\end{aligned}$$

.. and we are done! We factored another famous polynomial.

4.9.1 Exercises

1. Factor by recognizing the famous polynomial:

$$27v^3 + -216$$

2. Factor by recognizing the famous polynomial:

$$9w^2 + \frac{21w}{2} + \frac{49}{16}$$

3. Factor by recognizing the famous polynomial:

$$4t^2 + 4t + 1$$

4. Factor by recognizing the famous polynomial:

$$y^3 + -30y^2 + 300y + -1000$$

5. Factor by recognizing the famous polynomial:

$$4z^2 + 16z + 16$$

6. Factor by recognizing the famous polynomial:

$$\theta^3 + 6\theta^2 + 12\theta + 8$$

7. Factor by recognizing the famous polynomial:

$$9V^2 + 54V + 81$$

8. Factor by recognizing the famous polynomial:

$$4v^2 - 25$$

9. Factor by recognizing the famous polynomial:

$$x^2 + 2x + 1$$

10. Factor by recognizing the famous polynomial:

$$4\mu^2 + \frac{4\mu}{7} + \frac{1}{49}$$

11. Factor by recognizing the famous polynomial:

$$27\lambda^3 + -81\lambda^2 + 81\lambda + -27$$

12. Factor by recognizing the famous polynomial:

$$x^2 - 81$$

13. Factor by recognizing the famous polynomial:

$$8q^3 + 72q^2 + 216q + 216$$

14. Factor by recognizing the famous polynomial:

$$v^2 - 4$$

15. Factor by recognizing the famous polynomial:

$$9r^2 + 42r + 49$$

16. Factor by recognizing the famous polynomial:

$$X^2 + 10X + 25$$

17. Factor by recognizing the famous polynomial:

$$27w^3 + -135w^2 + 225w + -125$$

18. Factor by recognizing the famous polynomial:

$$8\omega^3 + 108\omega^2 + 486\omega + 729$$

19. Factor by recognizing the famous polynomial:

$$27\mu^3 + 216\mu^2 + 576\mu + 512$$

20. Factor by recognizing the famous polynomial:

$$9r^2 + 36r + 36$$

21. Factor by recognizing the famous polynomial:

$$X^3 + 1000$$

22. Factor by recognizing the famous polynomial:

$$V^2 + 4V + 4$$

23. Factor by recognizing the famous polynomial:

$$9Y^2 + 60Y + 100$$

24. Factor by recognizing the famous polynomial:

$$\xi^3 + 30\xi^2 + 300\xi + 1000$$

25. Factor by recognizing the famous polynomial:

$$27\rho^3 + -108\rho^2 + 144\rho + -64$$

26. Factor by recognizing the famous polynomial:

$$27\rho^3 + -512$$

27. Factor by recognizing the famous polynomial:

$$R^3 + 12R^2 + 48R + 64$$

28. Factor by recognizing the famous polynomial:

$$9u^2 + 60u + 100$$

29. Factor by recognizing the famous polynomial:

$$27W^3 + -54W^2 + 36W + -8$$

30. Factor by recognizing the famous polynomial:

$$4W^2 + \frac{16W}{9} + \frac{16}{81}$$

31. Factor by recognizing the famous polynomial:

$$9x^2 - 16$$

32. Factor by recognizing the famous polynomial:

$$9q^2 + \frac{14q}{3} + \frac{49}{81}$$

33. Factor by recognizing the famous polynomial:

$$9t^2 - 9$$

34. Factor by recognizing the famous polynomial:

$$w^3 + -216$$

35. Factor by recognizing the famous polynomial:

$$4r^2 - 64$$

4.10 Chapter Review

Very Famous Polynomials

name	Short	Example
FOIL	$[FOIL]$	$(A+B)(C+D) = AC + AD + BC + BD$
Difference of Squares	$[DS]$	$x^2 - y^2 = (x-y)(x+y)$
Difference of Cubes	$[DS]$	$x^3 - y^3 = (x-y)(x^2+xy+y^2)$
Sum of Cubes	$[SC]$	$x^3 + y^3 = (x+y)(x^2-xy+y^2)$
Sum of Cubes	$[PP2]$	$(x+y)^2 = x^2 + 2xy + y^2$

Pascal Polynomials

name	Short	Example
Pascal Polynomial #2	$[PP2]$	$(x+y)^2 = x^2 + 2yx + y^2$
Pascal Polynomial #3	$[PP3]$	$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
Pascal Polynomial #4	$[PP4]$	$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 6xy^3 + y^4$
Pascal Polynomial #5	$[PP5]$	$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Geometric Series Polynomials

name	Short	Example
Geometric Series #2	$[GS2]$	$x^2 - 1 = (x-1)(x+1)$
Geometric Series #3	$[GS3]$	$x^3 - 1 = (x-1)(x^2+x+1)$
Geometric Series #4	$[GS4]$	$x^4 - 1 = (x-1)(x^3+x^2+x+1)$
Geometric Series #5	$[GS5]$	$x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$

Generalized Geometric Series Polynomials

name	Short	Example
Geometric Series #2	$[GGS2]$	$x^2 - y^2 = (x-y)(x+y)$
Geometric Series #3	$[GGS3]$	$x^3 - y^3 = (x-y)(x^2+xy+y^2)$
Geometric Series #4	$[GGS4]$	$x^4 - y^4 = (x-1)(x^3+x^2y+xy^2+y^3)$
Geometric Series #5	$[GGS5]$	$x^5 - y^4 = (x-1)(x^4+x^3y+x^2y^2+xy^3+y^4)$

Some Answers

Section 4.1

1. $7r^6 + 2r^5 + 5r^4 + 5r^3 + 3r^2 + 2r + 5$

2. $2q^4 - 2q^3 + q^2 + 11q + 2$

3. $5w^6 + w^5 + 7w^4 - 5w^3 + w^2 - 10w + 7$

4. $5y + 2$

5. $3w^4 + 3w^3 - 5w^2 + 4w + 2$

6. $7r^3 - 2r^2 + 2$

7. $-3r^6 + 4r^5 + 5r^4 + 7r^3 - 5r^2 + 6r$

8. $8t^3 - 2t^2 + 10t + 4$

9. $-3y^5 - 6y^3 + 2y^2 - 4y - 2$

10. $6r^6 - 5r^5 + 5r^4 - 6r^3 - r^2 + 10r - 4$

11. $10r^4 + r^3 + 6r^2 + 7r + 7$

12. $9q - 7$

13. $-v^4 - 2v^3 - 3v^2 - 10v$

14. $6q^6 + q^5 - 12q^3 + 4q - 7$

15. $3t$

16. $-2z^5 - z^4 - 4z^3 + z^2 - 6z + 7$

17. $4v^4 + 6v^3 + 3v^2 - 3v + 4$

18. $6y^3 + 8y^2 + 2y + 2$

19. $-w - 1$

20. $13v^2 + 7v$

Section 4.2

1. try:

$$\begin{array}{r} u^4 + 6u^2 + 3u - 5 \\ \times \quad 7u^2 - 5u + 5 \\ \hline 5u^4 + 30u^2 + 15u - 25 \\ -5u^5 - 30u^3 - 15u^2 + 25u \\ 7u^6 + 42u^4 + 21u^3 - 35u^2 \\ \hline 7u^6 - 5u^5 + 47u^4 - 9u^3 - 20u^2 + 40u - 25 \end{array}$$

2. try:

$$\begin{array}{r} -5w^5 - 4w^3 + 2w \\ \times \quad 6w^6 \\ \hline -30w^{11} - 24w^9 + 12w^7 \end{array}$$

3. try:

$$\begin{aligned} & (-2z^7 + 3z^6 + 7z)(-2z^4) \\ =& (-2z^7)(-2z^4) + (3z^6)(-2z^4) + (7z)(-2z^4) && \text{(DL)} \\ =& 4z^{11} - 6z^{10} - 14z^5 && \text{(BI)} \end{aligned}$$

4. try:

$$\begin{array}{r} y^4 + 2y^3 + 7y^2 + 5y \\ \times \quad 5y^2 + 4y - 2 \\ \hline -2y^4 - 4y^3 - 14y^2 - 10y \\ 4y^5 + 8y^4 + 28y^3 + 20y^2 \\ 5y^6 + 10y^5 + 35y^4 + 25y^3 \\ \hline 5y^6 + 14y^5 + 41y^4 + 49y^3 + 6y^2 - 10y \end{array}$$

5. try:

$$\begin{array}{r} w^6 - 5w^4 + 7w^3 \\ \times \quad 7w^6 \\ \hline 7w^{12} - 35w^{10} + 49w^9 \end{array}$$

6. try:

$$\begin{array}{r} 4z^3 - 4z^2 - 5z + 7 \\ \times \quad 3z^2 - 2z - 4 \\ \hline -16z^3 + 16z^2 + 20z - 28 \\ -8z^4 + 8z^3 + 10z^2 - 14z \\ 12z^5 - 12z^4 - 15z^3 + 21z^2 \\ \hline 12z^5 - 20z^4 - 23z^3 + 47z^2 + 6z - 28 \end{array}$$

7. try:

$$\begin{array}{r} -w^3+5w^2-3w-4 \\ \times \quad 4-2w \\ \hline -4w^3+20w^2-12w-16 \\ 2w^4-10w^3+6w^2+8w \\ \hline 2w^4-14w^3+26w^2-4w-16 \end{array}$$

8. try:

$$\begin{array}{r} -4v^5+6v^3+3v^2 \\ \times \quad 3v^5 \\ \hline -12v^{10}+18v^8+9v^7 \end{array}$$

9. try:

$$\begin{array}{r} 3u^4+4u^2-2u-1 \\ \times \quad u^2-5u-2 \\ \hline -6u^4-8u^2+4u+2 \\ -15u^5-20u^3+10u^2+5u \\ 3u^6+4u^4-2u^3-u^2 \\ \hline 3u^6-15u^5-2u^4-22u^3+u^2+9u+2 \end{array}$$

10. try:

$$\begin{aligned} &(t^2+5t-5)(t^5-2t^2) \\ =&(t^2)(t^5-2t^2)+(5t)(t^5-2t^2)+(-5)(t^5-2t^2) && \text{(DL)} \\ =&(t^2)(t^5)+(t^2)(-2t^2)+(5t)(t^5)+(5t)(-2t^2)+(-5)(t^5)+(-5)(-2t^2) && \text{(DL)} \\ =&t^7+5t^6-5t^5-2t^4-10t^3+10t^2 && \text{(BI)} \end{aligned}$$

11. try:

$$\begin{array}{r} -4t^3-5t^2-t-2 \\ \times \quad 6t^2+3t \\ \hline -12t^4-15t^3-3t^2-6t \\ -24t^5-30t^4-6t^3-12t^2 \\ \hline -24t^5-42t^4-21t^3-15t^2-6t \end{array}$$

12. try:

$$\begin{array}{r} -3r^3+7r^2+1 \\ \times \quad 7r^3-5r^2 \\ \hline 15r^5-35r^4-5r^2 \\ -21r^6+49r^5+7r^3 \\ \hline -21r^6+64r^5-35r^4+7r^3-5r^2 \end{array}$$

13. try:

$$\begin{array}{r} -2z^2+z+4 \\ \times \quad 2z^3-2z \\ \hline 4z^3-2z^2-8z \\ -4z^5+2z^4+8z^3 \\ \hline -4z^5+2z^4+12z^3-2z^2-8z \end{array}$$

14. try:

$$\begin{array}{r} 4r^5+r^3-5r \\ \times \quad -r \\ \hline -4r^6-r^4+5r^2 \end{array}$$

15. try:

$$\begin{array}{r} 5w^4+7w^3+6w+3 \\ \times \quad -4w^2+4w+3 \\ \hline 15w^4+21w^3+18w+9 \\ 20w^5+28w^4+24w^2+12w \\ -20w^6-28w^5-24w^3-12w^2 \\ \hline -20w^6-8w^5+43w^4-3w^3+12w^2+30w+9 \end{array}$$

16. try:

$$\begin{aligned} &(-y^5-5y^3-3y^2)(5y) \\ =&(-y^5)(5y)+(-5y^3)(5y)+(-3y^2)(5y) && \text{(DL)} \\ =&-5y^6-25y^4-15y^3 && \text{(BI)} \end{aligned}$$

17. try:

$$\begin{aligned} &(6z^5+4z^3-1)(5z^7) \\ =&(6z^5)(5z^7)+(4z^3)(5z^7)+(-1)(5z^7) && \text{(DL)} \\ =&30z^{12}+20z^{10}-5z^7 && \text{(BI)} \end{aligned}$$

18. try:

$$\begin{array}{r} 6z^3 - 4z^2 - 5z \\ \times \quad 6z^3 - 2z \\ \hline -12z^4 + 8z^3 + 10z^2 \\ 36z^6 - 24z^5 - 30z^4 \\ \hline 36z^6 - 24z^5 - 42z^4 + 8z^3 + 10z^2 \end{array}$$

19. try:

$$(3y^2 + y - 4)(-3y^3 - 2y)$$

$$=(3y^2)(-3y^3 - 2y) + (y)(-3y^3 - 2y) + (-4)(-3y^3 - 2y) \quad \text{(DL)}$$

$$=(3y^2)(-3y^3) + (3y^2)(-2y) + (y)(-3y^3) + (y)(-2y) + (-4)(-3y^3) + (-4)(-2y) \quad \text{(DL)}$$

$$= -9y^5 - 3y^4 + 6y^3 - 2y^2 + 8y \quad \text{(BI)}$$

20. try:

$$\begin{array}{r} 6y^4 + y + 4 \\ \times \quad -4y^2 \\ \hline -24y^6 - 4y^3 - 16y^2 \end{array}$$

Section 4.3

1. vary, use DL or KG for example..

2.

$$(7w + 5)(5 - 4w) \quad \text{(given)}$$

$$= (7w)(-4w) + (7w)(5) + (5)(-4w) + (5)(5) \quad \text{(FOIL)}$$

$$= -28w^2 + 35w + -20w + 25 \quad \text{(BI)}$$

$$= -28w^2 + 15w + 25 \quad \text{(BI)}$$

3.

$$(3t + 2)(4t - 5) \quad \text{(given)}$$

$$= (3t)(4t) + (3t)(-5) + (2)(4t) + (2)(-5) \quad \text{(FOIL)}$$

$$= 12t^2 + -15t + 8t + -10 \quad \text{(BI)}$$

$$= 12t^2 - 7t - 10 \quad \text{(BI)}$$

4. using [PP2] ...

$$(v + 1)^2 \quad \text{(given)}$$

$$= (v)^2 + 2(1)(v) + (1)^2 \quad \text{(PP2)}$$

$$= v^2 + 2v + 1 \quad \text{(BI)}$$

5. using [PP3] ...

$$(w + y)^3 \quad \text{(given)}$$

$$(w + y)^2 (w + y)^1 \quad \text{(JAE)}$$

$$\left(w^2 + 2wy + y^2\right)(w + y) \quad \text{(PP2,BI)}$$

now we use KG

$$\begin{array}{r} w^2 + 2wy + y^2 \\ \times \quad w + y \\ \hline w^3 + 2w^2y + wy^2 \\ w^2y + 2wy^2 + y^3 \\ \hline w^3 + 3w^2y + 3wy^2 + y^3 \end{array}$$

6. this is famous [PP2] goes like this...

$$(\beta + 1)(\beta - 1) \quad \text{(given)}$$

$$= (\beta)(\beta) + (\beta)(-1) + (1)(\beta) + (1)(-1) \quad \text{(FOIL)}$$

$$= \beta^2 + -\beta + \beta + -1 \quad \text{(BI)}$$

$$= \beta^2 - 1 \quad \text{(BI)}$$

7.

$$(3q - 1)(3 - q) \quad \text{(given)}$$

$$= (3q)(-q) + (3q)(3) + (-1)(-q) + (-1)(3) \quad \text{(FOIL)}$$

$$= -3q^2 + 9q + q + -3 \quad \text{(BI)}$$

$$= -3q^2 + 10q - 3 \quad \text{(BI)}$$

8. using [PP2] ...

$$\left(2r^3 + 3\right)^2 \quad \text{(given)}$$

$$= \left(2r^3\right)^2 + 2(3)\left(2r^3\right) + (3)^2 \quad \text{(PP2)}$$

$$= 4r^6 + 12r^3 + 9 \quad \text{(BI)}$$

9.

$$
\begin{aligned}
&(5z+1)(7z+4) && \text{(given)}\\
&=(5z)(7z)+(5z)(4)+(1)(7z)+(1)(4) && \text{(FOIL)}\\
&=35z^2+20z+7z+4 && \text{(BI)}\\
&=35z^2+27z+4 && \text{(BI)}
\end{aligned}
$$

10. this is famous [DC] goes like this...

$$
\begin{aligned}
&\left(5u^2+u\right)\left(5u^2-u\right) && \text{(given)}\\
&=\left(5u^2\right)\left(5u^2\right)+\left(5u^2\right)(-u)+(u)\left(5u^2\right)+(u)(-u) && \text{(FOIL)}\\
&=25u^4+-5u^3+5u^3+-u^2 && \text{(BI)}\\
&=25u^4-u^2 && \text{(BI)}
\end{aligned}
$$

11. using [PP3] ...

$$
\begin{aligned}
&(r+z)^3 && \text{(given)}\\
&(r+z)^2(r+z)^1 && \text{(JAE)}\\
&\left(r^2+2rz+z^2\right)(r+z) && \text{(PP2,BI)}
\end{aligned}
$$

now we use KG

$$
\begin{array}{r}
r^2+2rz+z^2\\
\times \qquad r+z\\
\hline
r^3+2r^2z+rz^2\\
r^2z+2rz^2+z^3\\
\hline
r^3+3r^2z+3rz^2+z^3
\end{array}
$$

12. using [PP2] ...

$$
\begin{aligned}
&\left(4r^3+7r\right)^2 && \text{(given)}\\
&=\left(4r^3\right)^2+2(7r)\left(4r^3\right)+(7r)^2 && \text{(PP2)}\\
&=16r^6+56r^4+49r^2 && \text{(BI)}
\end{aligned}
$$

13. this is famous [DC] goes like this...

$$
\begin{aligned}
&(\theta+\phi)^2 && \text{(given)}\\
&(\theta+\phi)(\theta+\phi) && \text{(N.Expo)}\\
&=(\phi)(\phi)+(\phi)(\theta)+(\theta)(\phi)+(\theta)(\theta) && \text{(FOIL)}\\
&=\phi^2+\theta\phi+\theta\phi+\theta^2 && \text{(BI)}\\
&=\theta^2+2\theta\phi+\phi^2 && \text{(BI)}
\end{aligned}
$$

14. this is famous [DC] goes like this...

$$
\begin{aligned}
&(\beta+\phi)^2 && \text{(given)}\\
&(\beta+\phi)(\beta+\phi) && \text{(N.Expo)}\\
&=(\beta)(\beta)+(\beta)(\phi)+(\phi)(\beta)+(\phi)(\phi) && \text{(FOIL)}\\
&=\beta^2+\beta\phi+\beta\phi+\phi^2 && \text{(BI)}\\
&=\beta^2+2\beta\phi+\phi^2 && \text{(BI)}
\end{aligned}
$$

15.

$$
\begin{aligned}
&\left(6q^4-q^3\right)\left(7q^6-5q^7\right) && \text{(given)}\\
&=\left(6q^4\right)\left(-5q^7\right)+\left(6q^4\right)\left(7q^6\right)+\left(-q^3\right)\left(-5q^7\right)+\left(-q^3\right)\left(7q^6\right) && \text{(FOIL)}\\
&=-30q^{11}+42q^{10}+5q^{10}+-7q^9 && \text{(BI)}\\
&=-30q^{11}+47q^{10}-7q^9 && \text{(BI)}
\end{aligned}
$$

16. using [PP3] ...

$$
\begin{aligned}
&(4-y)^3 && \text{(given)}\\
&=(-y)^3+3(4)^2(-y)+3(4)(-y)^2+(4)^3 && \text{(PP2)}\\
&=-y^3+12y^2-48y+64 && \text{(BI)}
\end{aligned}
$$

17. using [PP3] ...

$$
\begin{aligned}
&(4-z)^3 && \text{(given)}\\
&=(-z)^3+3(4)^2(-z)+3(4)(-z)^2+(4)^3 && \text{(PP2)}\\
&=-z^3+12z^2-48z+64 && \text{(BI)}
\end{aligned}
$$

18. using [PP2] ...

$$
\begin{aligned}
&\left(7v^3+2\right)^2 && \text{(given)}\\
&=\left(7v^3\right)^2+2(2)\left(7v^3\right)+(2)^2 && \text{(PP2)}\\
&=49v^6+28v^3+4 && \text{(BI)}
\end{aligned}
$$

19. using [PP2] ...

$$
\begin{aligned}
&\left(3-4t^3\right)^2 && \text{(given)}\\
&=\left(-4t^3\right)^2+2(3)\left(-4t^3\right)+(3)^2 && \text{(PP2)}\\
&=16t^6-24t^3+9 && \text{(BI)}
\end{aligned}
$$

20. vary, use DL or KG for example..

21. using [PP2] ...

$$(3y+1)^2 \qquad \text{(given)}$$
$$= (3y)^2 + 2\,(1)\,(3y) + (1)^2 \qquad \text{(PP2)}$$
$$= 9y^2 + 6y + 1 \qquad \text{(BI)}$$

22.

$$(2v+5)\,(4-5v) \qquad \text{(given)}$$
$$= (2v)\,(-5v) + (2v)\,(4) + (5)\,(-5v) + (5)\,(4) \qquad \text{(FOIL)}$$
$$= -10v^2 + 8v + -25v + 20 \qquad \text{(BI)}$$
$$= -10v^2 - 17v + 20 \qquad \text{(BI)}$$

23. this is famous [DC] goes like this...

$$\left(2q^6 + 4q^2\right)\left(2q^6 - 4q^2\right) \qquad \text{(given)}$$
$$= \left(2q^6\right)\left(2q^6\right) + \left(2q^6\right)\left(-4q^2\right) + \left(4q^2\right)\left(2q^6\right) + \left(4q^2\right)\left(-4q^2\right) \qquad \text{(FOIL)}$$
$$= 4q^{12} + -8q^8 + 8q^8 + -16q^4 \qquad \text{(BI)}$$
$$= 4q^{12} - 16q^4 \qquad \text{(BI)}$$

24. this is famous [DC] goes like this...

$$\left(7z^5 - z^4\right)\left(7z^5 + z^4\right) \qquad \text{(given)}$$
$$= \left(7z^5\right)\left(7z^5\right) + \left(7z^5\right)\left(z^4\right) + \left(-z^4\right)\left(7z^5\right) + \left(-z^4\right)\left(z^4\right) \qquad \text{(FOIL)}$$
$$= 49z^{10} + 7z^9 + -7z^9 + -z^8 \qquad \text{(BI)}$$
$$= 49z^{10} - z^8 \qquad \text{(BI)}$$

25. this is famous [DC] goes like this...

$$\left(11z^5 + 4z^2\right)\left(11z^5 - 4z^2\right) \qquad \text{(given)}$$
$$= \left(11z^5\right)\left(11z^5\right) + \left(11z^5\right)\left(-4z^2\right) + \left(4z^2\right)\left(11z^5\right) + \left(4z^2\right)\left(-4z^2\right) \qquad \text{(FOIL)}$$
$$= 121z^{10} + -44z^7 + 44z^7 + -16z^4 \qquad \text{(BI)}$$
$$= 121z^{10} - 16z^4 \qquad \text{(BI)}$$

26.

$$(-4u-3)\,(-5u-3) \qquad \text{(given)}$$
$$= (-4u)\,(-5u) + (-4u)\,(-3) + (-3)\,(-5u) + (-3)\,(-3) \qquad \text{(FOIL)}$$
$$= 20u^2 + 12u + 15u + 9 \qquad \text{(BI)}$$
$$= 20u^2 + 27u + 9 \qquad \text{(BI)}$$

27. vary, use DL or KG for example..

28. this is famous [DC] goes like this...

$$(11v+5)\,(11v-5) \qquad \text{(given)}$$
$$= (11v)\,(11v) + (11v)\,(-5) + (5)\,(11v) + (5)\,(-5) \qquad \text{(FOIL)}$$
$$= 121v^2 + -55v + 55v + -25 \qquad \text{(BI)}$$
$$= 121v^2 - 25 \qquad \text{(BI)}$$

29. this is famous [DC] goes like this...

$$(\beta+\omega)^2 \qquad \text{(given)}$$
$$(\beta+\omega)\,(\beta+\omega) \qquad \text{(N.Expo)}$$
$$= (\omega)\,(\omega) + (\omega)\,(\beta) + (\beta)\,(\omega) + (\beta)\,(\beta) \qquad \text{(FOIL)}$$
$$= \omega^2 + \beta\omega + \beta\omega + \beta^2 \qquad \text{(BI)}$$
$$= \beta^2 + 2\beta\omega + \omega^2 \qquad \text{(BI)}$$

30.

$$\left(5y^5 - y^7\right)\left(7y^6 - y^7\right) \qquad \text{(given)}$$
$$= \left(-y^7\right)\left(-y^7\right) + \left(-y^7\right)\left(7y^6\right) + \left(5y^5\right)\left(-y^7\right) + \left(5y^5\right)\left(7y^6\right) \qquad \text{(FOIL)}$$
$$= y^{14} + -7y^{13} + -5y^{12} + 35y^{11} \qquad \text{(BI)}$$
$$= y^{14} - 7y^{13} - 5y^{12} + 35y^{11} \qquad \text{(BI)}$$

31. this is famous [PP2] goes like this...

$$(\mu+3)\,(\mu-3) \qquad \text{(given)}$$
$$= (\mu)\,(\mu) + (\mu)\,(-3) + (3)\,(\mu) + (3)\,(-3) \qquad \text{(FOIL)}$$
$$= \mu^2 + -3\mu + 3\mu + -9 \qquad \text{(BI)}$$
$$= \mu^2 - 9 \qquad \text{(BI)}$$

32. using [PP2] ...

$$\left(6q - 2q^3\right)^2 \qquad \text{(given)}$$
$$= \left(-2q^3\right)^2 + 2\,(6q)\left(-2q^3\right) + (6q)^2 \qquad \text{(PP2)}$$
$$= 4q^6 - 24q^4 + 36q^2 \qquad \text{(BI)}$$

33. this is famous [PP2] goes like this...

$$(\mu+1)\,(\mu-1) \qquad \text{(given)}$$
$$= (\mu)\,(\mu) + (\mu)\,(-1) + (1)\,(\mu) + (1)\,(-1) \qquad \text{(FOIL)}$$
$$= \mu^2 + -\mu + \mu + -1 \qquad \text{(BI)}$$
$$= \mu^2 - 1 \qquad \text{(BI)}$$

34.

$$(3-4r)(1-r) \qquad \text{(given)}$$

$$=(-4r)(-r)+(-4r)(1)+(3)(-r)+(3)(1) \qquad \text{(FOIL)}$$

$$=4r^2+-4r+-3r+3 \qquad \text{(BI)}$$

$$=4r^2-7r+3 \qquad \text{(BI)}$$

35. this is famous [DC] goes like this...

$$(\mu+\phi)(\phi-\mu) \qquad \text{(given)}$$

$$=(\phi)(\phi)+(\phi)(-\mu)+(\mu)(\phi)+(\mu)(-\mu) \qquad \text{(FOIL)}$$

$$=\phi^2+-\mu\phi+\mu\phi+-\mu^2 \qquad \text{(BI)}$$

$$=\phi^2-\mu^2 \qquad \text{(BI)}$$

Section 4.4

1.

$$\begin{array}{r|l}
 & 12y^3-16y^2-10y-6 \\ \hline
3y) & 36y^4-48y^3-30y^2-18y+27 \\
 & -36y^4 \\ \hline
 & -48y^3 \\
 & 48y^3 \\ \hline
 & -30y^2 \\
 & 30y^2 \\ \hline
 & -18y \\
 & 18y \\ \hline
\end{array}$$

thus final asnwers is

$$12y^3-16y^2-10y+\frac{9}{y}-6$$

2.

$$\begin{array}{r|l}
 & 24t^3-42t^2-30t+36 \\ \hline
-t) & -24t^4+42t^3+30t^2-36t+9 \\
 & 24t^4 \\ \hline
 & 42t^3 \\
 & -42t^3 \\ \hline
 & 30t^2 \\
 & -30t^2 \\ \hline
 & -36t \\
 & 36t \\ \hline
\end{array}$$

thus final asnwers is

$$24t^3-42t^2-30t-\frac{9}{t}+36$$

3.

$$-2q^9+8q^8-20q^2\div 3q^2 \qquad \text{(given)}$$

$$=\frac{-2q^9+8q^8-20q^2}{3q^2} \qquad \text{(def }\div\text{)}$$

$$=\frac{-2q^9}{3q^2}+\frac{8q^8}{3q^2}+\frac{-20q^2}{3q^2} \qquad \text{(ATT)}$$

$$=-\frac{2q^7}{3}+\frac{8q^6}{3}+-\frac{20}{3} \qquad \text{(BI)}$$

4.

$$\begin{array}{r|l}
 & 3u^3+\frac{51}{2}u^2+18u+12 \\ \hline
2u) & 6u^4+51u^3+36u^2+24u+54 \\
 & -6u^4 \\ \hline
 & 51u^3 \\
 & -51u^3 \\ \hline
 & 36u^2 \\
 & -36u^2 \\ \hline
 & 24u \\
 & -24u \\ \hline
\end{array}$$

thus final asnwers is

$$3u^3+\frac{51u^2}{2}+18u+\frac{27}{u}+12$$

5.

$$\begin{array}{r|l}
 & -51y^3-18y^2-36y+48 \\ \hline
-y) & 51y^4+18y^3+36y^2-48y-12 \\
 & -51y^4 \\ \hline
 & 18y^3 \\
 & -18y^3 \\ \hline
 & 36y^2 \\
 & -36y^2 \\ \hline
 & -48y \\
 & 48y \\ \hline
\end{array}$$

thus final asnwers is

$$-51y^3-18y^2-36y+\frac{12}{y}+48$$

6.

$$23w^8 - 2w^6 + 11w^5 \div -2w^3 \qquad \text{(given)}$$

$$= \frac{23w^8 - 2w^6 + 11w^5}{-2w^3} \qquad (\text{def } \div)$$

$$= \frac{23w^8}{-2w^3} + \frac{-2w^6}{-2w^3} + \frac{11w^5}{-2w^3} \qquad \text{(ATT)}$$

$$= -\frac{23w^5}{2} + w^3 + -\frac{11w^2}{2} \qquad \text{(BI)}$$

7.

$$\begin{array}{r|l}
 & \frac{3}{2}u^3 - \frac{39}{2}u^2 + \frac{33}{2}u - 15 \\ \hline
2u) & 3u^4 - 39u^3 + 33u^2 - 30u - 9 \\
 & -3u^4 \\ \hline
 & -39u^3 \\
 & 39u^3 \\ \hline
 & 33u^2 \\
 & -33u^2 \\ \hline
 & -30u \\
 & 30u \\ \hline
\end{array}$$

thus final asnwers is

$$\frac{3u^3}{2} - \frac{39u^2}{2} + \frac{33u}{2} - \frac{9}{2u} - 15$$

8.

$$-17y^6 - 22y^4 - 12y^3 \div -2y^3 \qquad \text{(given)}$$

$$= \frac{-17y^6 - 22y^4 - 12y^3}{-2y^3} \qquad (\text{def } \div)$$

$$= \frac{-17y^6}{-2y^3} + \frac{-22y^4}{-2y^3} + \frac{-12y^3}{-2y^3} \qquad \text{(ATT)}$$

$$= \frac{17y^3}{2} + 11y + 6 \qquad \text{(BI)}$$

9.

$$\begin{array}{r|l}
 & -6y^3 + 48y^2 + 60y + 39 \\ \hline
y) & -6y^4 + 48y^3 + 60y^2 + 39y - 33 \\
 & 6y^4 \\ \hline
 & 48y^3 \\
 & -48y^3 \\ \hline
 & 60y^2 \\
 & -60y^2 \\ \hline
 & 39y \\
 & -39y \\ \hline
\end{array}$$

thus final asnwers is

$$-6y^3 + 48y^2 + 60y - \frac{33}{y} + 39$$

10.

$$-26v^5 + 11v^4 + 14v^2 \div -2v^2 \qquad \text{(given)}$$

$$= \frac{-26v^5 + 11v^4 + 14v^2}{-2v^2} \qquad (\text{def } \div)$$

$$= \frac{-26v^5}{-2v^2} + \frac{11v^4}{-2v^2} + \frac{14v^2}{-2v^2} \qquad \text{(ATT)}$$

$$= 13v^3 + -\frac{11v^2}{2} + -7 \qquad \text{(BI)}$$

11.

$$\begin{array}{r|l}
 & -15y^3 + 4y^2 + 8y - 17 \\ \hline
-3y) & 45y^4 - 12y^3 - 24y^2 + 51y - 36 \\
 & -45y^4 \\ \hline
 & -12y^3 \\
 & 12y^3 \\ \hline
 & -24y^2 \\
 & 24y^2 \\ \hline
 & 51y \\
 & -51y \\ \hline
\end{array}$$

thus final asnwers is

$$-15y^3 + 4y^2 + 8y + \frac{12}{y} - 17$$

12.

$$-20v^7 + 15v^5 - 18v^3 \div -3v^2 \qquad \text{(given)}$$

$$= \frac{-20v^7 + 15v^5 - 18v^3}{-3v^2} \qquad (\text{def } \div)$$

$$= \frac{-20v^7}{-3v^2} + \frac{15v^5}{-3v^2} + \frac{-18v^3}{-3v^2} \qquad \text{(ATT)}$$

$$= \frac{20v^5}{3} + -5v^3 + 6v \qquad \text{(BI)}$$

13.

$$\begin{array}{rrrrrr}
 & & -6x^3 & +\frac{57}{2}x^2 & -\frac{21}{2}x & +15 \\ \cline{2-6}
-2x) & 12x^4 & -57x^3 & +21x^2 & -30x & +6 \\
 & -12x^4 & & & & \\ \cline{2-2}
 & & -57x^3 & & & \\
 & & 57x^3 & & & \\ \cline{3-3}
 & & & 21x^2 & & \\
 & & & -21x^2 & & \\ \cline{4-4}
 & & & & -30x & \\
 & & & & 30x & \\ \cline{5-5}
\end{array}$$

thus final asnwers is

$$-6x^3 + \frac{57x^2}{2} - \frac{21x}{2} - \frac{3}{x} + 15$$

14.

$$2z^8 - 6z^7 + 26z^6 \div 3z \qquad \text{(given)}$$

$$= \frac{2z^8 - 6z^7 + 26z^6}{3z} \qquad \text{(def } \div\text{)}$$

$$= \frac{2z^8}{3z} + \frac{-6z^7}{3z} + \frac{26z^6}{3z} \qquad \text{(ATT)}$$

$$= \frac{2z^7}{3} + -2z^6 + \frac{26z^5}{3} \qquad \text{(BI)}$$

15.

$$\begin{array}{rrrrrr}
 & & -11x^3 & +8x^2 & -10x & -19 \\ \cline{2-6}
3x) & -33x^4 & +24x^3 & -30x^2 & -57x & +42 \\
 & 33x^4 & & & & \\ \cline{2-2}
 & & 24x^3 & & & \\
 & & -24x^3 & & & \\ \cline{3-3}
 & & & -30x^2 & & \\
 & & & 30x^2 & & \\ \cline{4-4}
 & & & & -57x & \\
 & & & & 57x & \\ \cline{5-5}
\end{array}$$

thus final asnwers is

$$-11x^3 + 8x^2 - 10x + \frac{14}{x} - 19$$

16.

$$\begin{array}{rrrrrr}
 & & t^3 & -16t^2 & -9t & +6 \\ \cline{2-6}
-3t) & -3t^4 & +48t^3 & +27t^2 & -18t & +54 \\
 & 3t^4 & & & & \\ \cline{2-2}
 & & 48t^3 & & & \\
 & & -48t^3 & & & \\ \cline{3-3}
 & & & 27t^2 & & \\
 & & & -27t^2 & & \\ \cline{4-4}
 & & & & -18t & \\
 & & & & 18t & \\ \cline{5-5}
\end{array}$$

thus final asnwers is

$$t^3 - 16t^2 - 9t - \frac{18}{t} + 6$$

17.

$$\begin{array}{rrrrrr}
 & & -16y^3 & -19y^2 & +4y & -5 \\ \cline{2-5}
3y) & -48y^4 & -57y^3 & +12y^2 & -15y & -27 \\
 & 48y^4 & & & & \\ \cline{2-2}
 & & -57y^3 & & & \\
 & & 57y^3 & & & \\ \cline{3-3}
 & & & 12y^2 & & \\
 & & & -12y^2 & & \\ \cline{4-4}
 & & & & -15y & \\
 & & & & 15y & \\ \cline{5-5}
\end{array}$$

thus final asnwers is

$$-16y^3 - 19y^2 + 4y - \frac{9}{y} - 5$$

18.

$$-22r^{10} + 15r^9 - 12r^6 \div -2r^3 \qquad \text{(given)}$$

$$= \frac{-22r^{10} + 15r^9 - 12r^6}{-2r^3} \qquad \text{(def } \div\text{)}$$

$$= \frac{-22r^{10}}{-2r^3} + \frac{15r^9}{-2r^3} + \frac{-12r^6}{-2r^3} \qquad \text{(ATT)}$$

$$= 11r^7 + -\frac{15r^6}{2} + 6r^3 \qquad \text{(BI)}$$

19.

$$26t^7 + 13t^5 + 23t^3 \div -3t \qquad \text{(given)}$$

$$= \frac{26t^7 + 13t^5 + 23t^3}{-3t} \qquad \text{(def } \div\text{)}$$

$$= \frac{26t^7}{-3t} + \frac{13t^5}{-3t} + \frac{23t^3}{-3t} \qquad \text{(ATT)}$$

$$= -\frac{26t^6}{3} + -\frac{13t^4}{3} + -\frac{23t^2}{3} \qquad \text{(BI)}$$

20.

$$\begin{array}{r} 15u^3 + 33u^2 - 42u - 36 \\ -u \overline{)\, -15u^4 - 33u^3 + 42u^2 + 36u + 60} \\ \underline{15u^4} \qquad\qquad\qquad\qquad\qquad\quad \\ -33u^3 \qquad\qquad\qquad\quad \\ \underline{33u^3} \qquad\qquad\qquad\quad \\ 42u^2 \qquad\qquad\quad \\ \underline{-42u^2} \qquad\qquad\quad \\ 36u \qquad\quad \\ \underline{-36u} \qquad\quad \end{array}$$

thus final asnwers is

$$15u^3 + 33u^2 - 42u - \frac{60}{u} - 36$$

21.

$$- v^9 + 19v^7 - 14v^2 \div -v^3 \qquad \text{(given)}$$

$$= \frac{-v^9 + 19v^7 - 14v^2}{-v^3} \qquad \text{(def } \div\text{)}$$

$$= \frac{-v^9}{-v^3} + \frac{19v^7}{-v^3} + \frac{-14v^2}{-v^3} \qquad \text{(ATT)}$$

$$= v^6 + -19v^4 + \frac{14}{v} \qquad \text{(BI)}$$

22.

$$\begin{array}{r} -33u^3 + 27u^2 + 30u + 60 \\ -u \overline{)\, 33u^4 - 27u^3 - 30u^2 - 60u - 54} \\ \underline{-33u^4} \qquad\qquad\qquad\qquad\qquad\quad \\ -27u^3 \qquad\qquad\qquad\quad \\ \underline{27u^3} \qquad\qquad\qquad\quad \\ -30u^2 \qquad\qquad\quad \\ \underline{30u^2} \qquad\qquad\quad \\ -60u \qquad\quad \\ \underline{60u} \qquad\quad \end{array}$$

thus final asnwers is

$$-33u^3 + 27u^2 + 30u + \frac{54}{u} + 60$$

23.

$$\begin{array}{r} -\frac{27}{2}y^3 - 21y^2 + 9y + 15 \\ 2y \overline{)\, -27y^4 - 42y^3 + 18y^2 + 30y + 21} \\ \underline{27y^4} \qquad\qquad\qquad\qquad\qquad\quad \\ -42y^3 \qquad\qquad\qquad\quad \\ \underline{42y^3} \qquad\qquad\qquad\quad \\ 18y^2 \qquad\qquad\quad \\ \underline{-18y^2} \qquad\qquad\quad \\ 30y \qquad\quad \\ \underline{-30y} \qquad\quad \end{array}$$

thus final asnwers is

$$-\frac{27y^3}{2} - 21y^2 + 9y + \frac{21}{2y} + 15$$

24.

$$- 11y^8 + 3y^5 - 22y^2 \div 3y^3 \qquad \text{(given)}$$

$$= \frac{-11y^8 + 3y^5 - 22y^2}{3y^3} \qquad \text{(def } \div\text{)}$$

$$= \frac{-11y^8}{3y^3} + \frac{3y^5}{3y^3} + \frac{-22y^2}{3y^3} \qquad \text{(ATT)}$$

$$= -\frac{11y^5}{3} + y^2 + -\frac{22}{3y} \qquad \text{(BI)}$$

25.

$$\begin{array}{r} -\frac{15}{2}x^3 - \frac{21}{2}x^2 + \frac{39}{2}x + \frac{9}{2} \\ -2x \overline{)\, 15x^4 + 21x^3 - 39x^2 - 9x - 3} \\ \underline{-15x^4} \qquad\qquad\qquad\qquad\qquad\quad \\ 21x^3 \qquad\qquad\qquad\quad \\ \underline{-21x^3} \qquad\qquad\qquad\quad \\ -39x^2 \qquad\qquad\quad \\ \underline{39x^2} \qquad\qquad\quad \\ -9x \qquad\quad \\ \underline{9x} \qquad\quad \end{array}$$

thus final asnwers is

$$-\frac{15x^3}{2} - \frac{21x^2}{2} + \frac{39x}{2} + \frac{3}{2x} + \frac{9}{2}$$

Section 4.5

1.

$$\begin{array}{r} -12t + 2 \\ 8t - 2 \overline{)\, -96t^2 + 40t - 1} \\ \underline{96t^2 - 24t} \quad\;\; \\ 16t - 1 \\ \underline{-16t + 4} \\ 3 \end{array}$$

thus final asnwers is

$$2 - 12t + \frac{3}{8t - 2}$$

2.

$$\begin{array}{r l}
 & 12x^4 - 8x^3 - 4x^2 + x - 7 \\ \hline
-7x^2 + x + 6\,) & -84x^6 + 68x^5 + 92x^4 - 59x^3 + 26x^2 - x - 50 \\
 & 84x^6 - 12x^5 - 72x^4 \\ \hline
 & 56x^5 + 20x^4 - 59x^3 \\
 & -56x^5 + 8x^4 + 48x^3 \\ \hline
 & 28x^4 - 11x^3 + 26x^2 \\
 & -28x^4 + 4x^3 + 24x^2 \\ \hline
 & -7x^3 + 50x^2 - x \\
 & 7x^3 - x^2 - 6x \\ \hline
 & 49x^2 - 7x - 50 \\
 & -49x^2 + 7x + 42 \\ \hline
 & -8
\end{array}$$

thus final asnwers is

$$12x^4 - 8x^3 - 4x^2 + x - 7 + -\frac{8}{-7x^2 + x + 6}$$

3.

$$\begin{array}{r l}
 & -5y^4 - 6y^3 - 8y^2 + y - 2 \\ \hline
-6y^2 - 8y + 3\,) & 30y^6 + 76y^5 + 81y^4 + 40y^3 - 20y^2 + 19y - 9 \\
 & -30y^6 - 40y^5 + 15y^4 \\ \hline
 & 36y^5 + 96y^4 + 40y^3 \\
 & -36y^5 - 48y^4 + 18y^3 \\ \hline
 & 48y^4 + 58y^3 - 20y^2 \\
 & -48y^4 - 64y^3 + 24y^2 \\ \hline
 & -6y^3 + 4y^2 + 19y \\
 & 6y^3 + 8y^2 - 3y \\ \hline
 & 12y^2 + 16y - 9 \\
 & -12y^2 - 16y + 6 \\ \hline
 & -3
\end{array}$$

thus final asnwers is

$$-5y^4 - 6y^3 - 8y^2 + y - 2 + -\frac{3}{-6y^2 - 8y + 3}$$

4.

$$\begin{array}{r l}
 & -10y^2 - 2y - 3 \\ \hline
-9y - 3\,) & 90y^3 + 48y^2 + 33y + 12 \\
 & -90y^3 - 30y^2 \\ \hline
 & 18y^2 + 33y \\
 & -18y^2 - 6y \\ \hline
 & 27y + 12 \\
 & -27y - 9 \\ \hline
 & 3
\end{array}$$

thus final asnwers is

$$-10y^2 - 2y - 3 + \frac{3}{-9y - 3}$$

5.

$$\begin{array}{r l}
 & -2y^3 + 8y^2 - 7 \\ \hline
6y - 11\,) & -12y^4 + 70y^3 - 88y^2 - 42y + 71 \\
 & 12y^4 - 22y^3 \\ \hline
 & 48y^3 - 88y^2 \\
 & -48y^3 + 88y^2 \\ \hline
 & -42y + 71 \\
 & 42y - 77 \\ \hline
 & -6
\end{array}$$

thus final asnwers is

$$-2y^3 + 8y^2 - 7 + -\frac{6}{6y - 11}$$

6.

$$\begin{array}{r l}
 & -2x^4 + x^3 - 11x^2 - 4x + 8 \\ \hline
10x + 5\,) & -20x^5 - 105x^3 - 95x^2 + 60x + 31 \\
 & 20x^5 + 10x^4 \\ \hline
 & 10x^4 - 105x^3 \\
 & -10x^4 - 5x^3 \\ \hline
 & -110x^3 - 95x^2 \\
 & 110x^3 + 55x^2 \\ \hline
 & -40x^2 + 60x \\
 & 40x^2 + 20x \\ \hline
 & 80x + 31 \\
 & -80x - 40 \\ \hline
 & -9
\end{array}$$

thus final asnwers is

$$-2x^4 + x^3 - 11x^2 - 4x + 8 + -\frac{9}{10x + 5}$$

7.

$$\begin{array}{r l}
 & -11u^4 - u^2 + 2u - 12 \\ \hline
-12u - 8\,) & 132u^5 + 88u^4 + 12u^3 - 16u^2 + 128u + 98 \\
 & -132u^5 - 88u^4 \\ \hline
 & 12u^3 - 16u^2 \\
 & -12u^3 - 8u^2 \\ \hline
 & -24u^2 + 128u \\
 & 24u^2 + 16u \\ \hline
 & 144u + 98 \\
 & -144u - 96 \\ \hline
 & 2
\end{array}$$

thus final asnwers is

$$-11u^4 - u^2 + 2u - 12 + \frac{2}{-12u - 8}$$

8.

$$\begin{array}{r l}
 & 10y^3 + 7y^2 - 2y + 4 \\ \hline
4y - 10\,) & 40y^4 - 72y^3 - 78y^2 + 36y - 34 \\
 & -40y^4 + 100y^3 \\ \hline
 & 28y^3 - 78y^2 \\
 & -28y^3 + 70y^2 \\ \hline
 & -8y^2 + 36y \\
 & 8y^2 - 20y \\ \hline
 & 16y - 34 \\
 & -16y + 40 \\ \hline
 & 6
\end{array}$$

thus final asnwers is

$$10y^3 + 7y^2 - 2y + 4 + \frac{6}{4y - 10}$$

9.

$$\begin{array}{r}
3u\ -6 \\ \hline
-3u-4)\ -9u^2\ +6u+23 \\
9u^2+12u \\ \hline
18u+23 \\
-18u-24 \\ \hline
-1
\end{array}$$

thus final asnwers is

$$3u-6+-\frac{1}{-3u-4}$$

10.

$$\begin{array}{r}
-5y^4\quad -6y^2+12y\ +9 \\ \hline
-y+2)\ \ 5y^5-10y^4+6y^3-24y^2+15y+25 \\
-5y^5+10y^4 \\ \hline
6y^3-24y^2 \\
-6y^3+12y^2 \\ \hline
-12y^2+15y \\
12y^2-24y \\ \hline
-9y+25 \\
9y-18 \\ \hline
7
\end{array}$$

thus final asnwers is

$$-5y^4-6y^2+12y+9+\frac{7}{2-y}$$

11.

$$\begin{array}{r}
-11y^4\ -6y^3\ -8y^2\ +3y\ +9 \\ \hline
-9y-2)\ \ 99y^5+76y^4+84y^3-11y^2-87y-25 \\
-99y^5-22y^4 \\ \hline
54y^4+84y^3 \\
-54y^4-12y^3 \\ \hline
72y^3-11y^2 \\
-72y^3-16y^2 \\ \hline
-27y^2-87y \\
27y^2\ +6y \\ \hline
-81y-25 \\
81y+18 \\ \hline
-7
\end{array}$$

thus final asnwers is

$$-11y^4-6y^3-8y^2+3y+9+-\frac{7}{-9y-2}$$

12.

$$\begin{array}{r}
7y^4\quad +5y^2\ +8y\ +4 \\ \hline
-4y+9)\ -28y^5+63y^4-20y^3+13y^2+56y+28 \\
28y^5-63y^4 \\ \hline
-20y^3+13y^2 \\
20y^3-45y^2 \\ \hline
-32y^2+56y \\
32y^2-72y \\ \hline
-16y+28 \\
16y-36 \\ \hline
-8
\end{array}$$

thus final asnwers is

$$7y^4+5y^2+8y+4+-\frac{8}{9-4y}$$

13.

$$\begin{array}{r}
4u^3\ -2u^2\ +12u\ -8 \\ \hline
-6u-12)\ -24u^4-36u^3-48u^2\ -96u+88 \\
24u^4+48u^3 \\ \hline
12u^3-48u^2 \\
-12u^3-24u^2 \\ \hline
-72u^2\ -96u \\
72u^2+144u \\ \hline
48u+88 \\
-48u-96 \\ \hline
-8
\end{array}$$

thus final asnwers is

$$4u^3-2u^2+12u-8+-\frac{8}{-6u-12}$$

14.

$$\begin{array}{r}
-11y\ +7 \\ \hline
-9y^2+3)\ \ 99y^3-63y^2-33y+17 \\
-99y^3\qquad +33y \\ \hline
-63y^2\qquad +17 \\
63y^2\qquad -21 \\ \hline
-4
\end{array}$$

thus final asnwers is

$$7-11y+-\frac{4}{3-9y^2}$$

15.

$$\begin{array}{r}
-11u^4\ -5u^3\ +7u^2\ +3u\ -4 \\ \hline
5u-11)\ -55u^5\ +96u^4+90u^3-62u^2-53u+45 \\
55u^5-121u^4 \\ \hline
-25u^4+90u^3 \\
25u^4-55u^3 \\ \hline
35u^3-62u^2 \\
-35u^3+77u^2 \\ \hline
15u^2-53u \\
-15u^2+33u \\ \hline
-20u+45 \\
20u-44 \\ \hline
1
\end{array}$$

thus final asnwers is

$$-11u^4-5u^3+7u^2+3u-4+\frac{1}{5u-11}$$

Section 4.6

1.

$$\begin{aligned} & 8t^{11} - 11t^6 - 7t^3 && \text{(given)} \\ &= -7\left(t^3\right) + 8\left(t^3\right)t^8 - 11\left(t^3\right)t^3 && \text{(Bi, factor gcd)} \\ &= \left(8t^8 - 11t^3 - 7\right)\left(t^3\right) && \text{(DL, BI)} \end{aligned}$$

2.

$$\begin{aligned} & 3u^5 - 5u^4 && \text{(given)} \\ &= 3\left(u^4\right)u - 5\left(u^4\right) && \text{(Bi, factor gcd)} \\ &= (3u - 5)\left(u^4\right) && \text{(DL, BI)} \end{aligned}$$

3.

$$\begin{aligned} & 9t^{12} + 7t^7 + 8t^4 && \text{(given)} \\ &= 9t^8\left(t^4\right) + 7t^3\left(t^4\right) + 8\left(t^4\right) && \text{(Bi, factor gcd)} \\ &= \left(9t^8 + 7t^3 + 8\right)\left(t^4\right) && \text{(DL, BI)} \end{aligned}$$

4.

$$\begin{aligned} & -8t^6 + 11t^3 + 10t && \text{(given)} \\ &= 10(t) - 8(t)t^5 + 11(t)t^2 && \text{(Bi, factor gcd)} \\ &= \left(-8t^5 + 11t^2 + 10\right)(t) && \text{(DL, BI)} \end{aligned}$$

5.

$$\begin{aligned} & 12y^7 - 7y^4 - 8y^2 && \text{(given)} \\ &= 12\left(y^2\right)y^5 - 8\left(y^2\right) - 7\left(y^2\right)y^2 && \text{(Bi, factor gcd)} \\ &= \left(12y^5 - 7y^2 - 8\right)\left(y^2\right) && \text{(DL, BI)} \end{aligned}$$

6.

$$\begin{aligned} & -5\left(t^3\right) - 9\left(t^3\right)u && \text{(given)} \\ &= (-9u - 5)\left(t^3\right) && \text{(DL, BI)} \end{aligned}$$

7.

$$\begin{aligned} & 11u^5\left(u^3\right) - 12u^2\left(u^3\right) + 7\left(u^3\right) && \text{(given)} \\ &= \left(11u^5 - 12u^2 + 7\right)\left(u^3\right) && \text{(DL, BI)} \end{aligned}$$

8.

$$\begin{aligned} & 9t^8 + 5t^5 - 4t^3 && \text{(given)} \\ &= -4\left(t^3\right) + 9\left(t^3\right)t^5 + 5\left(t^3\right)t^2 && \text{(Bi, factor gcd)} \\ &= \left(9t^5 + 5t^2 - 4\right)\left(t^3\right) && \text{(DL, BI)} \end{aligned}$$

9.

$$\begin{aligned} & -6y^{12} + 4y^7 + 7y^4 && \text{(given)} \\ &= 7\left(y^4\right) - 6\left(y^4\right)y^8 + 4\left(y^4\right)y^3 && \text{(Bi, factor gcd)} \\ &= \left(-6y^8 + 4y^3 + 7\right)\left(y^4\right) && \text{(DL, BI)} \end{aligned}$$

10.

$$\begin{aligned} & 12t^5 - 2t^2 + 7t && \text{(given)} \\ &= 7(t) + 12(t)t^4 - 2(t)t && \text{(Bi, factor gcd)} \\ &= \left(12t^4 - 2t + 7\right)(t) && \text{(DL, BI)} \end{aligned}$$

11.

$$\begin{aligned} & 12\left(x^3 + 8\right) + 8\left(x^3 + 8\right)u^5 - \left(x^3 + 8\right)u^2 && \text{(given)} \\ &= \left(8u^5 - u^2 + 12\right)\left(x^3 + 8\right) && \text{(DL, BI)} \end{aligned}$$

12.

$$\begin{aligned} & t^{12} + 12t^7 + 8t^4 && \text{(given)} \\ &= 8\left(t^4\right) + \left(t^4\right)t^8 + 12\left(t^4\right)t^3 && \text{(Bi, factor gcd)} \\ &= \left(t^8 + 12t^3 + 8\right)\left(t^4\right) && \text{(DL, BI)} \end{aligned}$$

13.

$$3(5x+12)u-8(5x+12) \qquad \text{(given)}$$

$$=(3u-8)\,(5x+12) \qquad \text{(DL, BI)}$$

14.

$$7t^8+3t^5-8t^3 \qquad \text{(given)}$$

$$=7t^5\left(t^3\right)+3t^2\left(t^3\right)-8\left(t^3\right) \qquad \text{(Bi, factor gcd)}$$

$$=\left(7t^5+3t^2-8\right)\left(t^3\right) \qquad \text{(DL, BI)}$$

15.

$$5u^2\omega^2+u\omega^2-8\omega^2 \qquad \text{(given)}$$

$$=\left(5u^2+u-8\right)\left(\omega^2\right) \qquad \text{(DL, BI)}$$

16.

$$y^4-12y^5 \qquad \text{(given)}$$

$$=\left(y^4\right)-12\left(y^4\right)y \qquad \text{(Bi, factor gcd)}$$

$$=(1-12y)\left(y^4\right) \qquad \text{(DL, BI)}$$

17.

$$7t^2-9t \qquad \text{(given)}$$

$$=7(t)t-9(t) \qquad \text{(Bi, factor gcd)}$$

$$=(7t-9)\,(t) \qquad \text{(DL, BI)}$$

18.

$$10\mu^4+8\mu^4x^3-12\mu^4x^2+3\mu^4x \qquad \text{(given)}$$

$$=\left(8x^3-12x^2+3x+10\right)\left(\mu^4\right) \qquad \text{(DL, BI)}$$

19.

$$11t^8(11u+2)-7t^3(11u+2)-(11u+2) \qquad \text{(given)}$$

$$=\left(11t^8-7t^3-1\right)(11u+2) \qquad \text{(DL, BI)}$$

20.

$$7t^5\left(x^3\right)-2t^2\left(x^3\right)-11\left(x^3\right) \qquad \text{(given)}$$

$$=\left(7t^5-2t^2-11\right)\left(x^3\right) \qquad \text{(DL, BI)}$$

21.

$$2\left(x^3\right)-4\left(x^3\right)t^4+\left(x^3\right)t \qquad \text{(given)}$$

$$=\left(-4t^4+t+2\right)\left(x^3\right) \qquad \text{(DL, BI)}$$

22.

$$-7\lambda^4-8\lambda^4x \qquad \text{(given)}$$

$$=(-8x-7)\left(\lambda^4\right) \qquad \text{(DL, BI)}$$

23.

$$-11\theta^2-2\theta^2x \qquad \text{(given)}$$

$$=(-2x-11)\left(\theta^2\right) \qquad \text{(DL, BI)}$$

24.

$$-7(7y+11)+4(7y+11)t^8+6(7y+11)t^3 \qquad \text{(given)}$$

$$=\left(4t^8+6t^3-7\right)(7y+11) \qquad \text{(DL, BI)}$$

25.

$$-4\mu^2-6\mu^2t^2-3\mu^2t \qquad \text{(given)}$$

$$=\left(-6t^2-3t-4\right)\left(\mu^2\right) \qquad \text{(DL, BI)}$$

26.

$$11x^2+4x \qquad \text{(given)}$$

$$=4(x)+11(x)x \qquad \text{(Bi, factor gcd)}$$

$$=(11x+4)\,(x) \qquad \text{(DL, BI)}$$

27.

$$
\begin{aligned}
& -10u^7 + 9u^4 + 11u^2 && \text{(given)} \\
&= 11\left(u^2\right) - 10\left(u^2\right)u^5 + 9\left(u^2\right)u^2 && \text{(Bi, factor gcd)} \\
&= \left(-10u^5 + 9u^2 + 11\right)\left(u^2\right) && \text{(DL, BI)}
\end{aligned}
$$

28.

$$
\begin{aligned}
& 11u^4 + 6u^3 && \text{(given)} \\
&= 6\left(u^3\right) + 11\left(u^3\right)u && \text{(Bi, factor gcd)} \\
&= (11u + 6)\left(u^3\right) && \text{(DL, BI)}
\end{aligned}
$$

29.

$$
\begin{aligned}
& 6u^4\left(x^6\right) - 9u\left(x^6\right) - 10\left(x^6\right) && \text{(given)} \\
&= \left(6u^4 - 9u - 10\right)\left(x^6\right) && \text{(DL, BI)}
\end{aligned}
$$

30.

$$
\begin{aligned}
& 12\left(y^2\right) + 11\left(y^2\right)x^5 + 9\left(y^2\right)x^2 && \text{(given)} \\
&= \left(11x^5 + 9x^2 + 12\right)\left(y^2\right) && \text{(DL, BI)}
\end{aligned}
$$

Section 4.7

1.

$$
\begin{aligned}
& 2 + -4x^2 + -4x^3 + 8x^5 && \text{(given)} \\
&= \left(2 + -4x^2\right) + \left(-4x^3 + 8x^5\right) && \text{(ALA)} \\
&= \left[-2(-1) + 4x^2(-1)\right] + \left[-2\left(2x^3\right) + 4x^2\left(2x^3\right)\right] && \text{(BI)} \\
&= \left(-2 + 4x^2\right)(-1) + \left(-2 + 4x^2\right)\left(2x^3\right) && \text{(DL)} \\
&= \left(-2 + 4x^2\right)\left(-1 + 2x^3\right) && \text{(DL)}
\end{aligned}
$$

2.

$$
\begin{aligned}
& 12 + 4\rho^2 + 3\rho^3 + \rho^5 && \text{(given)} \\
&= \left(12 + 4\rho^2\right) + \left(3\rho^3 + \rho^5\right) && \text{(ALA)} \\
&= \left[3(4) + \rho^2(4)\right] + \left[3\left(\rho^3\right) + \rho^2\left(\rho^3\right)\right] && \text{(BI)} \\
&= \left(3 + \rho^2\right)(4) + \left(3 + \rho^2\right)\left(\rho^3\right) && \text{(DL)} \\
&= \left(3 + \rho^2\right)\left(4 + \rho^3\right) && \text{(DL)}
\end{aligned}
$$

3.

$$
\begin{aligned}
& 1 + -t^3 + -t^2 + t^5 && \text{(given)} \\
&= 1 + -t^2 + -t^3 + t^5 && \text{(ColA, ALA)} \\
&= \left(1 + -t^2\right) + \left(-t^3 + t^5\right) && \text{(ALA)} \\
&= \left[-1(-1) + t^2(-1)\right] + \left[-1\left(t^3\right) + t^2\left(t^3\right)\right] && \text{(BI)} \\
&= \left(-1 + t^2\right)(-1) + \left(-1 + t^2\right)\left(t^3\right) && \text{(DL)} \\
&= \left(-1 + t^2\right)\left(-1 + t^3\right) && \text{(DL)}
\end{aligned}
$$

4.

$$
\begin{aligned}
& 4 + 3Q^2 + 4Q^3 + 3Q^5 && \text{(given)} \\
&= \left(4 + 3Q^2\right) + \left(4Q^3 + 3Q^5\right) && \text{(ALA)} \\
&= \left[4(1) + 3Q^2(1)\right] + \left[4\left(Q^3\right) + 3Q^2\left(Q^3\right)\right] && \text{(BI)} \\
&= \left(4 + 3Q^2\right)(1) + \left(4 + 3Q^2\right)\left(Q^3\right) && \text{(DL)} \\
&= \left(4 + 3Q^2\right)\left(1 + Q^3\right) && \text{(DL)}
\end{aligned}
$$

5.

$$
\begin{aligned}
& 20 + 8w + 5w^2 + 2w^3 && \text{(given)}\\
&= (20 + 8w) + \left(5w^2 + 2w^3\right) && \text{(ALA)}\\
&= [5\,(4) + 2w\,(4)] + \left[5\left(w^2\right) + 2w\left(w^2\right)\right] && \text{(BI)}\\
&= (5 + 2w)\,(4) + (5 + 2w)\left(w^2\right) && \text{(DL)}\\
&= (5 + 2w)\left(4 + w^2\right) && \text{(DL)}
\end{aligned}
$$

6.

$$
\begin{aligned}
& 2 + 10x + 2x^2 + 10x^3 && \text{(given)}\\
&= (2 + 10x) + \left(2x^2 + 10x^3\right) && \text{(ALA)}\\
&= [1\,(2) + 5x\,(2)] + \left[1\left(2x^2\right) + 5x\left(2x^2\right)\right] && \text{(BI)}\\
&= (1 + 5x)\,(2) + (1 + 5x)\left(2x^2\right) && \text{(DL)}\\
&= (1 + 5x)\left(2 + 2x^2\right) && \text{(DL)}
\end{aligned}
$$

7.

$$
\begin{aligned}
& 6 + 10v^2 + 6v^3 + 10v^5 && \text{(given)}\\
&= \left(6 + 10v^2\right) + \left(6v^3 + 10v^5\right) && \text{(ALA)}\\
&= \left[3\,(2) + 5v^2\,(2)\right] + \left[3\left(2v^3\right) + 5v^2\left(2v^3\right)\right] && \text{(BI)}\\
&= \left(3 + 5v^2\right)(2) + \left(3 + 5v^2\right)\left(2v^3\right) && \text{(DL)}\\
&= \left(3 + 5v^2\right)\left(2 + 2v^3\right) && \text{(DL)}
\end{aligned}
$$

8.

$$
\begin{aligned}
& 6 + 3v + 10v^2 + 5v^3 && \text{(given)}\\
&= (6 + 3v) + \left(10v^2 + 5v^3\right) && \text{(ALA)}\\
&= [2\,(3) + v\,(3)] + \left[2\left(5v^2\right) + v\left(5v^2\right)\right] && \text{(BI)}\\
&= (2 + v)\,(3) + (2 + v)\left(5v^2\right) && \text{(DL)}\\
&= (2 + v)\left(3 + 5v^2\right) && \text{(DL)}
\end{aligned}
$$

9.

$$
\begin{aligned}
& 16 + 12\theta^2 + 16\theta^3 + 12\theta^5 && \text{(given)}\\
&= \left(16 + 12\theta^2\right) + \left(16\theta^3 + 12\theta^5\right) && \text{(ALA)}\\
&= \left[4\,(4) + 3\theta^2\,(4)\right] + \left[4\left(4\theta^3\right) + 3\theta^2\left(4\theta^3\right)\right] && \text{(BI)}\\
&= \left(4 + 3\theta^2\right)(4) + \left(4 + 3\theta^2\right)\left(4\theta^3\right) && \text{(DL)}\\
&= \left(4 + 3\theta^2\right)\left(4 + 4\theta^3\right) && \text{(DL)}
\end{aligned}
$$

10.

$$
\begin{aligned}
& -2 + 5T^2 + -8T^3 + 20T^5 && \text{(given)}\\
&= \left(-2 + 5T^2\right) + \left(-8T^3 + 20T^5\right) && \text{(ALA)}\\
&= \left[-2\,(1) + 5T^2\,(1)\right] + \left[-2\left(4T^3\right) + 5T^2\left(4T^3\right)\right] && \text{(BI)}\\
&= \left(-2 + 5T^2\right)(1) + \left(-2 + 5T^2\right)\left(4T^3\right) && \text{(DL)}\\
&= \left(-2 + 5T^2\right)\left(1 + 4T^3\right) && \text{(DL)}
\end{aligned}
$$

11.

$$10 + 20z + 10z^2 + 20z^3 \quad \text{(given)}$$
$$= (10 + 20z) + \left(10z^2 + 20z^3\right) \quad \text{(ALA)}$$
$$= [2(5) + 4z(5)] + \left[2\left(5z^2\right) + 4z\left(5z^2\right)\right] \quad \text{(BI)}$$
$$= (2 + 4z)(5) + (2 + 4z)\left(5z^2\right) \quad \text{(DL)}$$
$$= (2 + 4z)\left(5 + 5z^2\right) \quad \text{(DL)}$$

12.

$$15 + 3T^2 + 15T^3 + 3T^5 \quad \text{(given)}$$
$$= \left(15 + 3T^2\right) + \left(15T^3 + 3T^5\right) \quad \text{(ALA)}$$
$$= \left[5(3) + T^2(3)\right] + \left[5\left(3T^3\right) + T^2\left(3T^3\right)\right] \quad \text{(BI)}$$
$$= \left(5 + T^2\right)(3) + \left(5 + T^2\right)\left(3T^3\right) \quad \text{(DL)}$$
$$= \left(5 + T^2\right)\left(3 + 3T^3\right) \quad \text{(DL)}$$

13.

$$3 + 12x^2 + 4x^3 + 16x^5 \quad \text{(given)}$$
$$= \left(3 + 12x^2\right) + \left(4x^3 + 16x^5\right) \quad \text{(ALA)}$$
$$= \left[1(3) + 4x^2(3)\right] + \left[1\left(4x^3\right) + 4x^2\left(4x^3\right)\right] \quad \text{(BI)}$$
$$= \left(1 + 4x^2\right)(3) + \left(1 + 4x^2\right)\left(4x^3\right) \quad \text{(DL)}$$
$$= \left(1 + 4x^2\right)\left(3 + 4x^3\right) \quad \text{(DL)}$$

14.

$$4 + -10\beta^3 + -6\beta^2 + 15\beta^5 \quad \text{(given)}$$
$$=4 + -6\beta^2 + -10\beta^3 + 15\beta^5 \quad \text{(ColA, ALA)}$$
$$= \left(4 + -6\beta^2\right) + \left(-10\beta^3 + 15\beta^5\right) \quad \text{(ALA)}$$
$$= \left[-2(-2) + 3\beta^2(-2)\right] + \left[-2\left(5\beta^3\right) + 3\beta^2\left(5\beta^3\right)\right] \quad \text{(BI)}$$
$$= \left(-2 + 3\beta^2\right)(-2) + \left(-2 + 3\beta^2\right)\left(5\beta^3\right) \quad \text{(DL)}$$
$$= \left(-2 + 3\beta^2\right)\left(-2 + 5\beta^3\right) \quad \text{(DL)}$$

15.

$$3 + 15\lambda^3 + 5\lambda^2 + 25\lambda^5 \quad \text{(given)}$$
$$=3 + 5\lambda^2 + 15\lambda^3 + 25\lambda^5 \quad \text{(ColA, ALA)}$$
$$= \left(3 + 5\lambda^2\right) + \left(15\lambda^3 + 25\lambda^5\right) \quad \text{(ALA)}$$
$$= \left[3(1) + 5\lambda^2(1)\right] + \left[3\left(5\lambda^3\right) + 5\lambda^2\left(5\lambda^3\right)\right] \quad \text{(BI)}$$
$$= \left(3 + 5\lambda^2\right)(1) + \left(3 + 5\lambda^2\right)\left(5\lambda^3\right) \quad \text{(DL)}$$
$$= \left(3 + 5\lambda^2\right)\left(1 + 5\lambda^3\right) \quad \text{(DL)}$$

16.

$$12 + 12\xi^3 + 6\xi^2 + 6\xi^5 \quad \text{(given)}$$
$$=12 + 6\xi^2 + 12\xi^3 + 6\xi^5 \quad \text{(ColA, ALA)}$$
$$= \left(12 + 6\xi^2\right) + \left(12\xi^3 + 6\xi^5\right) \quad \text{(ALA)}$$
$$= \left[4(3) + 2\xi^2(3)\right] + \left[4\left(3\xi^3\right) + 2\xi^2\left(3\xi^3\right)\right] \quad \text{(BI)}$$
$$= \left(4 + 2\xi^2\right)(3) + \left(4 + 2\xi^2\right)\left(3\xi^3\right) \quad \text{(DL)}$$
$$= \left(4 + 2\xi^2\right)\left(3 + 3\xi^3\right) \quad \text{(DL)}$$

17.

$$
\begin{aligned}
& 20 + 15q + 16q^2 + 12q^3 && \text{(given)} \\
&= (20 + 15q) + \left(16q^2 + 12q^3\right) && \text{(ALA)} \\
&= [4\,(5) + 3q\,(5)] + \left[4\left(4q^2\right) + 3q\left(4q^2\right)\right] && \text{(BI)} \\
&= (4 + 3q)\,(5) + (4 + 3q)\left(4q^2\right) && \text{(DL)} \\
&= (4 + 3q)\left(5 + 4q^2\right) && \text{(DL)}
\end{aligned}
$$

18.

$$
\begin{aligned}
& 12 + 4\phi^2 + 12\phi^3 + 4\phi^5 && \text{(given)} \\
&= \left(12 + 4\phi^2\right) + \left(12\phi^3 + 4\phi^5\right) && \text{(ALA)} \\
&= \left[3\,(4) + \phi^2\,(4)\right] + \left[3\left(4\phi^3\right) + \phi^2\left(4\phi^3\right)\right] && \text{(BI)} \\
&= \left(3 + \phi^2\right)(4) + \left(3 + \phi^2\right)\left(4\phi^3\right) && \text{(DL)} \\
&= \left(3 + \phi^2\right)\left(4 + 4\phi^3\right) && \text{(DL)}
\end{aligned}
$$

19.

$$
\begin{aligned}
& -1 + 2\lambda^2 + -4\lambda^3 + 8\lambda^5 && \text{(given)} \\
&= \left(-1 + 2\lambda^2\right) + \left(-4\lambda^3 + 8\lambda^5\right) && \text{(ALA)} \\
&= \left[-1\,(1) + 2\lambda^2\,(1)\right] + \left[-1\left(4\lambda^3\right) + 2\lambda^2\left(4\lambda^3\right)\right] && \text{(BI)} \\
&= \left(-1 + 2\lambda^2\right)(1) + \left(-1 + 2\lambda^2\right)\left(4\lambda^3\right) && \text{(DL)} \\
&= \left(-1 + 2\lambda^2\right)\left(1 + 4\lambda^3\right) && \text{(DL)}
\end{aligned}
$$

20.

$$
\begin{aligned}
& -1 + -2U^3 + 4U^2 + 8U^5 && \text{(given)} \\
&= -1 + 4U^2 + -2U^3 + 8U^5 && \text{(ColA, ALA)} \\
&= \left(-1 + 4U^2\right) + \left(-2U^3 + 8U^5\right) && \text{(ALA)} \\
&= \left[-1\,(1) + 4U^2\,(1)\right] + \left[-1\left(2U^3\right) + 4U^2\left(2U^3\right)\right] && \text{(BI)} \\
&= \left(-1 + 4U^2\right)(1) + \left(-1 + 4U^2\right)\left(2U^3\right) && \text{(DL)} \\
&= \left(-1 + 4U^2\right)\left(1 + 2U^3\right) && \text{(DL)}
\end{aligned}
$$

Section 4.8

1. First take the product of the outside terms, $4\lambda^2 \cdot 24 = 96\lambda^2$ Then we look at all the way we can factor this coefficent, $96 \mapsto$ as: $\pm[1, 96]$, $\pm[2, 48]$, $\pm[3, 32]$, $\pm[4, 24]$, $\pm[6, 16]$, $\pm[8, 12]$ We scan for a suitable pair to split 20λ into. Idenfiying $20\lambda = 8\lambda + 12\lambda$ as a suitable candidate. Then we stop talking start the doing!

$$
\begin{aligned}
& 4\lambda^2 + 20\lambda + 24 && \text{(given)} \\
&= 4\lambda^2 + (8\lambda + 12\lambda) + 24 && \text{(Bi, The Split!)} \\
&= \left(4\lambda^2 + 8\lambda\right) + (12\lambda + 24) && \text{(ALA)} \\
&= (2\lambda + 4)\,2\lambda + (2\lambda + 4)\,6 && \text{(DL, BI)} \\
&= (2\lambda + 4)\,(2\lambda + 6) && \text{(DL)}
\end{aligned}
$$

2. First take the product of the outside terms, $6x^4 \cdot 60 = 360x^4$ Then we look at all the way we can factor this coefficent, $360 \mapsto$ as: $\pm[1, 360]$, $\pm[2, 180]$, $\pm[3, 120]$, $\pm[4, 90]$, $\pm[5, 72]$, $\pm[6, 60]$, $\pm[8, 45]$, $\pm[9, 40]$, $\pm[10, 36]$, $\pm[12, 30]$, $\pm[15, 24]$, $\pm[18, 20]$ We scan for a suitable pair to split $42x^2$ into. Idenfiying $42x^2 = 30x^2 + 12x^2$ as a suitable candidate. Then we stop talking start the doing!

$$6x^4 + 42x^2 + 60 \qquad \text{(given)}$$

$$=6x^4 + \left(30x^2 + 12x^2\right) + 60 \qquad \text{(Bi, The Split!)}$$

$$= \left(6x^4 + 30x^2\right) + \left(12x^2 + 60\right) \qquad \text{(ALA)}$$

$$= \left(2x^2 + 10\right) 3x^2 + \left(2x^2 + 10\right) 6 \qquad \text{(DL, BI)}$$

$$= \left(2x^2 + 10\right) \left(3x^2 + 6\right) \qquad \text{(DL)}$$

3. First take the product of the outside terms, $z^2 \cdot 4 = 4z^2$ Then we look at all the way we can factor this coefficent, $4 \mapsto$ as: $\pm[1, 4]$, $\pm[2, 2]$ We scan for a suitable pair to split $-4z$ into. Idenfiying $-4z = -2z + -2z$ as a suitable candidate. Then we stop talking start the doing!

$$z^2 + -4z + 4 \qquad \text{(given)}$$

$$=z^2 + (-2z + -2z) + 4 \qquad \text{(Bi, The Split!)}$$

$$= \left(z^2 + -2z\right) + (-2z + 4) \qquad \text{(ALA)}$$

$$= (z + -2)\, z + (z + -2) - 2 \qquad \text{(DL, BI)}$$

$$= (z + -2)\,(z + -2) \qquad \text{(DL)}$$

4. First take the product of the outside terms, $6\beta^2 \cdot 8 = 48\beta^2$ Then we look at all the way we can factor this coefficent, $48 \mapsto$ as: $\pm[1, 48]$, $\pm[2, 24]$, $\pm[3, 16]$, $\pm[4, 12]$, $\pm[6, 8]$ We scan for a suitable pair to split 26β into. Idenfiying $26\beta = 2\beta + 24\beta$ as a suitable candidate. Then we stop talking start the doing!

$$6\beta^2 + 26\beta + 8 \qquad \text{(given)}$$

$$=6\beta^2 + (2\beta + 24\beta) + 8 \qquad \text{(Bi, The Split!)}$$

$$= \left(6\beta^2 + 2\beta\right) + (24\beta + 8) \qquad \text{(ALA)}$$

$$= (3\beta + 1)\, 2\beta + (3\beta + 1)\, 8 \qquad \text{(DL, BI)}$$

$$= (3\beta + 1)\,(2\beta + 8) \qquad \text{(DL)}$$

5. First take the product of the outside terms, $6\mu^6 \cdot 36z^4 = 216\mu^6 z^4$ Then we look at all the way we can factor this coefficent, $216 \mapsto$ as: $\pm[1, 216]$, $\pm[2, 108]$, $\pm[3, 72]$, $\pm[4, 54]$, $\pm[6, 36]$, $\pm[8, 27]$, $\pm[9, 24]$, $\pm[12, 18]$ We scan for a suitable pair to split $30\mu^3 z^2$ into. Idenfiying $30\mu^3 z^2 = 12\mu^3 z^2 + 18\mu^3 z^2$ as a suitable candidate. Then we stop talking start the doing!

$$6\mu^6 + 30\mu^3 z^2 + 36z^4 \qquad \text{(given)}$$

$$=6\mu^6 + \left(12\mu^3 z^2 + 18\mu^3 z^2\right) + 36z^4 \qquad \text{(Bi, The Split!)}$$

$$= \left(6\mu^6 + 12\mu^3 z^2\right) + \left(18\mu^3 z^2 + 36z^4\right) \qquad \text{(ALA)}$$

$$= \left(2\mu^3 + 4z^2\right) 3\mu^3 + \left(2\mu^3 + 4z^2\right) 9z^2 \qquad \text{(DL, BI)}$$

$$= \left(2\mu^3 + 4z^2\right) \left(3\mu^3 + 9z^2\right) \qquad \text{(DL)}$$

6. First take the product of the outside terms, $2\rho^6 \cdot 30r^4 = 60\rho^6 r^4$ Then we look at all the way we can factor this coefficent, $60 \mapsto$ as: $\pm[1, 60]$, $\pm[2, 30]$, $\pm[3, 20]$, $\pm[4, 15]$, $\pm[5, 12]$, $\pm[6, 10]$ We scan for a suitable pair to split $16\rho^3 r^2$ into. Idenfiying $16\rho^3 r^2 = 6\rho^3 r^2 + 10\rho^3 r^2$ as a suitable candidate. Then we stop talking start the doing!

$$2\rho^6 + 16\rho^3 r^2 + 30r^4 \qquad \text{(given)}$$

$$=2\rho^6 + \left(6\rho^3 r^2 + 10\rho^3 r^2\right) + 30r^4 \qquad \text{(Bi, The Split!)}$$

$$= \left(2\rho^6 + 6\rho^3 r^2\right) + \left(10\rho^3 r^2 + 30r^4\right) \qquad \text{(ALA)}$$

$$= \left(2\rho^3 + 6r^2\right) \rho^3 + \left(2\rho^3 + 6r^2\right) 5r^2 \qquad \text{(DL, BI)}$$

$$= \left(2\rho^3 + 6r^2\right) \left(\rho^3 + 5r^2\right) \qquad \text{(DL)}$$

7. First take the product of the outside terms, $2q^2 \cdot 55 = 110q^2$ Then we look at all the way we can factor this coefficent, $110 \mapsto$ as: $\pm[1, 110]$, $\pm[2, 55]$, $\pm[5, 22]$, $\pm[10, 11]$ We scan for a suitable pair to split $27q$ into. Idenfiying $27q = 22q + 5q$ as a suitable candidate. Then we stop talking start the doing!

$$2q^2 + 27q + 55 \qquad \text{(given)}$$

$$=2q^2 + (22q + 5q) + 55 \qquad \text{(Bi, The Split!)}$$

$$= \left(2q^2 + 22q\right) + (5q + 55) \qquad \text{(ALA)}$$

$$= (q + 11)\, 2q + (q + 11)\, 5 \qquad \text{(DL, BI)}$$

$$= (q + 11)\,(2q + 5) \qquad \text{(DL)}$$

8. First take the product of the outside terms, $2x^4 \cdot 45 = 90x^4$ Then we look at all the way we can factor this coefficent, $90 \mapsto$ as: $\pm[1, 90]$, $\pm[2, 45]$, $\pm[3, 30]$, $\pm[5, 18]$, $\pm[6, 15]$, $\pm[9, 10]$ We scan for a suitable pair to split $23x^2$ into. Idenfiying $23x^2 = 18x^2 + 5x^2$ as a suitable candidate. Then we stop talking start the doing!

$$
\begin{aligned}
& 2x^4 + 23x^2 + 45 && \text{(given)}\\
=& 2x^4 + \left(18x^2 + 5x^2\right) + 45 && \text{(Bi, The Split!)}\\
=& \left(2x^4 + 18x^2\right) + \left(5x^2 + 45\right) && \text{(ALA)}\\
=& \left(x^2 + 9\right) 2x^2 + \left(x^2 + 9\right) 5 && \text{(DL, BI)}\\
=& \left(x^2 + 9\right) \left(2x^2 + 5\right) && \text{(DL)}
\end{aligned}
$$

9. First take the product of the outside terms, $3\gamma^6 \cdot 30q^4 = 90\gamma^6 q^4$ Then we look at all the way we can factor this coefficent, $90 \mapsto$ as: $\pm[1, 90]$, $\pm[2, 45]$, $\pm[3, 30]$, $\pm[5, 18]$, $\pm[6, 15]$, $\pm[9, 10]$ We scan for a suitable pair to split $21\gamma^3 q^2$ into. Idenfiying $21\gamma^3 q^2 = 6\gamma^3 q^2 + 15\gamma^3 q^2$ as a suitable candidate. Then we stop talking start the doing!

$$
\begin{aligned}
& 3\gamma^6 + 21\gamma^3 q^2 + 30q^4 && \text{(given)}\\
=& 3\gamma^6 + \left(6\gamma^3 q^2 + 15\gamma^3 q^2\right) + 30q^4 && \text{(Bi, The Split!)}\\
=& \left(3\gamma^6 + 6\gamma^3 q^2\right) + \left(15\gamma^3 q^2 + 30q^4\right) && \text{(ALA)}\\
=& \left(3\gamma^3 + 6q^2\right) \gamma^3 + \left(3\gamma^3 + 6q^2\right) 5q^2 && \text{(DL, BI)}\\
=& \left(3\gamma^3 + 6q^2\right) \left(\gamma^3 + 5q^2\right) && \text{(DL)}
\end{aligned}
$$

10. First take the product of the outside terms, $t^2 \cdot 84 = 84t^2$ Then we look at all the way we can factor this coefficent, $84 \mapsto$ as: $\pm[1, 84]$, $\pm[2, 42]$, $\pm[3, 28]$, $\pm[4, 21]$, $\pm[6, 14]$, $\pm[7, 12]$ We scan for a suitable pair to split $19t$ into. Idenfiying $19t = 7t + 12t$ as a suitable candidate. Then we stop talking start the doing!

$$
\begin{aligned}
& t^2 + 19t + 84 && \text{(given)}\\
=& t^2 + (7t + 12t) + 84 && \text{(Bi, The Split!)}\\
=& \left(t^2 + 7t\right) + (12t + 84) && \text{(ALA)}\\
=& (t + 7)\, t + (t + 7)\, 12 && \text{(DL, BI)}\\
=& (t + 7)\, (t + 12) && \text{(DL)}
\end{aligned}
$$

11. First take the product of the outside terms, $3\mu^6 \cdot 3t^4 = 9\mu^6 t^4$ Then we look at all the way we can factor this coefficent, $9 \mapsto$ as: $\pm[1, 9]$, $\pm[3, 3]$ We scan for a suitable pair to split $10\mu^3 t^2$ into. Idenfiying $10\mu^3 t^2 = \mu^3 t^2 + 9\mu^3 t^2$ as a suitable candidate. Then we stop talking start the doing!

$$
\begin{aligned}
& 3\mu^6 + 10\mu^3 t^2 + 3t^4 && \text{(given)}\\
=& 3\mu^6 + \left(\mu^3 t^2 + 9\mu^3 t^2\right) + 3t^4 && \text{(Bi, The Split!)}\\
=& \left(3\mu^6 + \mu^3 t^2\right) + \left(9\mu^3 t^2 + 3t^4\right) && \text{(ALA)}\\
=& \left(3\mu^3 + t^2\right) \mu^3 + \left(3\mu^3 + t^2\right) 3t^2 && \text{(DL, BI)}\\
=& \left(3\mu^3 + t^2\right) \left(\mu^3 + 3t^2\right) && \text{(DL)}
\end{aligned}
$$

12. First take the product of the outside terms, $6x^2 \cdot -120 = -720x^2$ Then we look at all the way we can factor this coefficent, $-720 \mapsto$ as: $\pm[1, 720]$, $\pm[2, 360]$, $\pm[3, 240]$, $\pm[4, 180]$, $\pm[5, 144]$, $\pm[6, 120]$, $\pm[8, 90]$, $\pm[9, 80]$, $\pm[10, 72]$, $\pm[12, 60]$, $\pm[15, 48]$, $\pm[16, 45]$, $\pm[18, 40]$, $\pm[20, 36]$, $\pm[24, 30]$ We scan for a suitable pair to split $-6x$ into. Idenfiying $-6x = 24x + -30x$ as a suitable candidate. Then we stop talking start the doing!

$$
\begin{aligned}
& 6x^2 + -6x + -120 && \text{(given)}\\
=& 6x^2 + (24x + -30x) + -120 && \text{(Bi, The Split!)}\\
=& \left(6x^2 + 24x\right) + (-30x + -120) && \text{(ALA)}\\
=& (3x + 12)\, 2x + (3x + 12) - 10 && \text{(DL, BI)}\\
=& (3x + 12)\, (2x + -10) && \text{(DL)}
\end{aligned}
$$

13. First take the product of the outside terms, $2\lambda^2 \cdot 21 = 42\lambda^2$ Then we look at all the way we can factor this coefficent, $42 \mapsto$ as: $\pm[1,42]$, $\pm[2,21]$, $\pm[3,14]$, $\pm[6,7]$ We scan for a suitable pair to split 17λ into. Idenfiying $17\lambda = 14\lambda + 3\lambda$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned} & 2\lambda^2 + 17\lambda + 21 && \text{(given)} \\ =& 2\lambda^2 + (14\lambda + 3\lambda) + 21 && \text{(Bi, The Split!)} \\ =& \left(2\lambda^2 + 14\lambda\right) + (3\lambda + 21) && \text{(ALA)} \\ =& (\lambda + 7)\, 2\lambda + (\lambda + 7)\, 3 && \text{(DL, BI)} \\ =& (\lambda + 7)\,(2\lambda + 3) && \text{(DL)} \end{aligned}$$

14. First take the product of the outside terms, $2x^2 \cdot 121 = 242x^2$ Then we look at all the way we can factor this coefficent, $242 \mapsto$ as: $\pm[1,242]$, $\pm[2,121]$, $\pm[11,22]$ We scan for a suitable pair to split $33x$ into. Idenfiying $33x = 22x + 11x$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned} & 2x^2 + 33x + 121 && \text{(given)} \\ =& 2x^2 + (22x + 11x) + 121 && \text{(Bi, The Split!)} \\ =& \left(2x^2 + 22x\right) + (11x + 121) && \text{(ALA)} \\ =& (x + 11)\, 2x + (x + 11)\, 11 && \text{(DL, BI)} \\ =& (x + 11)\,(2x + 11) && \text{(DL)} \end{aligned}$$

15. First take the product of the outside terms, $-w^2 \cdot 12 = -12w^2$ Then we look at all the way we can factor this coefficent, $-12 \mapsto$ as: $\pm[1,12]$, $\pm[2,6]$, $\pm[3,4]$ We scan for a suitable pair to split $11w$ into. Idenfiying $11w = 12w + -w$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned} & -w^2 + 11w + 12 && \text{(given)} \\ =& -w^2 + (12w + -w) + 12 && \text{(Bi, The Split!)} \\ =& \left(-w^2 + 12w\right) + (-w + 12) && \text{(ALA)} \\ =& (w + -12) - w + (w + -12) - 1 && \text{(DL, BI)} \\ =& (w + -12)\,(-w + -1) && \text{(DL)} \end{aligned}$$

16. First take the product of the outside terms, $-2v^2 \cdot 3 = -6v^2$ Then we look at all the way we can factor this coefficent, $-6 \mapsto$ as: $\pm[1,6]$, $\pm[2,3]$ We scan for a suitable pair to split $5v$ into. Idenfiying $5v = 6v + -v$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned} & -2v^2 + 5v + 3 && \text{(given)} \\ =& -2v^2 + (6v + -v) + 3 && \text{(Bi, The Split!)} \\ =& \left(-2v^2 + 6v\right) + (-v + 3) && \text{(ALA)} \\ =& (v + -3) - 2v + (v + -3) - 1 && \text{(DL, BI)} \\ =& (v + -3)\,(-2v + -1) && \text{(DL)} \end{aligned}$$

17. First take the product of the outside terms, $4q^2 \cdot 8 = 32q^2$ Then we look at all the way we can factor this coefficent, $32 \mapsto$ as: $\pm[1,32]$, $\pm[2,16]$, $\pm[4,8]$ We scan for a suitable pair to split $12q$ into. Idenfiying $12q = 4q + 8q$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned} & 4q^2 + 12q + 8 && \text{(given)} \\ =& 4q^2 + (4q + 8q) + 8 && \text{(Bi, The Split!)} \\ =& \left(4q^2 + 4q\right) + (8q + 8) && \text{(ALA)} \\ =& (2q + 2)\, 2q + (2q + 2)\, 4 && \text{(DL, BI)} \\ =& (2q + 2)\,(2q + 4) && \text{(DL)} \end{aligned}$$

18. First take the product of the outside terms, $2w^2 \cdot 60 = 120w^2$ Then we look at all the way we can factor this coefficent, $120 \mapsto$ as: $\pm[1,120]$, $\pm[2,60]$, $\pm[3,40]$, $\pm[4,30]$, $\pm[5,24]$, $\pm[6,20]$, $\pm[8,15]$, $\pm[10,12]$ We scan for a suitable pair to split $22w$ into. Idenfiying $22w = 10w + 12w$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned} & 2w^2 + 22w + 60 && \text{(given)} \\ =& 2w^2 + (10w + 12w) + 60 && \text{(Bi, The Split!)} \\ =& \left(2w^2 + 10w\right) + (12w + 60) && \text{(ALA)} \\ =& (w + 5)\, 2w + (w + 5)\, 12 && \text{(DL, BI)} \\ =& (w + 5)\,(2w + 12) && \text{(DL)} \end{aligned}$$

19. First take the product of the outside terms, $6r^4 \cdot 35 = 210r^4$ Then we look at all the way we can factor this coefficent, $210 \mapsto$ as: $\pm[1,210]$, $\pm[2,105]$, $\pm[3,70]$, $\pm[5,42]$, $\pm[6,35]$, $\pm[7,30]$, $\pm[10,21]$, $\pm[14,15]$ We scan for a suitable pair to split $31r^2$ into. Idenfiying $31r^2 = 21r^2 + 10r^2$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned}
&6r^4 + 31r^2 + 35 && \text{(given)}\\
=&6r^4 + \left(21r^2 + 10r^2\right) + 35 && \text{(Bi, The Split!)}\\
=&\left(6r^4 + 21r^2\right) + \left(10r^2 + 35\right) && \text{(ALA)}\\
=&\left(2r^2 + 7\right) 3r^2 + \left(2r^2 + 7\right) 5 && \text{(DL, BI)}\\
=&\left(2r^2 + 7\right)\left(3r^2 + 5\right) && \text{(DL)}
\end{aligned}$$

20. First take the product of the outside terms, $3\omega^6 \cdot 40t^4 = 120t^4\omega^6$ Then we look at all the way we can factor this coefficent, $120 \mapsto$ as: $\pm[1,120]$, $\pm[2,60]$, $\pm[3,40]$, $\pm[4,30]$, $\pm[5,24]$, $\pm[6,20]$, $\pm[8,15]$, $\pm[10,12]$ We scan for a suitable pair to split $23t^2\omega^3$ into. Idenfiying $23t^2\omega^3 = 8t^2\omega^3 + 15t^2\omega^3$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned}
&3\omega^6 + 23t^2\omega^3 + 40t^4 && \text{(given)}\\
=&3\omega^6 + \left(8t^2\omega^3 + 15t^2\omega^3\right) + 40t^4 && \text{(Bi, The Split!)}\\
=&\left(3\omega^6 + 8t^2\omega^3\right) + \left(15t^2\omega^3 + 40t^4\right) && \text{(ALA)}\\
=&\left(3\omega^3 + 8t^2\right)\omega^3 + \left(3\omega^3 + 8t^2\right)5t^2 && \text{(DL, BI)}\\
=&\left(3\omega^3 + 8t^2\right)\left(\omega^3 + 5t^2\right) && \text{(DL)}
\end{aligned}$$

21. First take the product of the outside terms, $2\beta^2 \cdot 42 = 84\beta^2$ Then we look at all the way we can factor this coefficent, $84 \mapsto$ as: $\pm[1,84]$, $\pm[2,42]$, $\pm[3,28]$, $\pm[4,21]$, $\pm[6,14]$, $\pm[7,12]$ We scan for a suitable pair to split 19β into. Idenfiying $19\beta = 7\beta + 12\beta$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned}
&2\beta^2 + 19\beta + 42 && \text{(given)}\\
=&2\beta^2 + (7\beta + 12\beta) + 42 && \text{(Bi, The Split!)}\\
=&\left(2\beta^2 + 7\beta\right) + (12\beta + 42) && \text{(ALA)}\\
=&(2\beta + 7)\,\beta + (2\beta + 7)\,6 && \text{(DL, BI)}\\
=&(2\beta + 7)(\beta + 6) && \text{(DL)}
\end{aligned}$$

22. First take the product of the outside terms, $3\lambda^2 \cdot 6 = 18\lambda^2$ Then we look at all the way we can factor this coefficent, $18 \mapsto$ as: $\pm[1,18]$, $\pm[2,9]$, $\pm[3,6]$ We scan for a suitable pair to split 19λ into. Idenfiying $19\lambda = 18\lambda + \lambda$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned}
&3\lambda^2 + 19\lambda + 6 && \text{(given)}\\
=&3\lambda^2 + (18\lambda + \lambda) + 6 && \text{(Bi, The Split!)}\\
=&\left(3\lambda^2 + 18\lambda\right) + (\lambda + 6) && \text{(ALA)}\\
=&(\lambda + 6)\,3\lambda + (\lambda + 6)\,1 && \text{(DL, BI)}\\
=&(\lambda + 6)(3\lambda + 1) && \text{(DL)}
\end{aligned}$$

23. First take the product of the outside terms, $2t^4 \cdot 72 = 144t^4$ Then we look at all the way we can factor this coefficent, $144 \mapsto$ as: $\pm[1,144]$, $\pm[2,72]$, $\pm[3,48]$, $\pm[4,36]$, $\pm[6,24]$, $\pm[8,18]$, $\pm[9,16]$, $\pm[12,12]$ We scan for a suitable pair to split $25t^2$ into. Idenfiying $25t^2 = 16t^2 + 9t^2$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned}
&2t^4 + 25t^2 + 72 && \text{(given)}\\
=&2t^4 + \left(16t^2 + 9t^2\right) + 72 && \text{(Bi, The Split!)}\\
=&\left(2t^4 + 16t^2\right) + \left(9t^2 + 72\right) && \text{(ALA)}\\
=&\left(t^2 + 8\right)2t^2 + \left(t^2 + 8\right)9 && \text{(DL, BI)}\\
=&\left(t^2 + 8\right)\left(2t^2 + 9\right) && \text{(DL)}
\end{aligned}$$

24. First take the product of the outside terms, $2z^2 \cdot -12 = -24z^2$ Then we look at all the way we can factor this coefficent, $-24 \mapsto$ as: $\pm[1,24]$, $\pm[2,12]$, $\pm[3,8]$, $\pm[4,6]$ We scan for a suitable pair to split $-2z$ into. Idenfiying $-2z = 4z + -6z$ as a suitable candidate. Then we stop talking start the doing!

$$\begin{aligned}
&2z^2 + -2z + -12 && \text{(given)}\\
=&2z^2 + (4z + -6z) + -12 && \text{(Bi, The Split!)}\\
=&\left(2z^2 + 4z\right) + (-6z + -12) && \text{(ALA)}\\
=&(z + 2)\,2z + (z + 2) - 6 && \text{(DL, BI)}\\
=&(z + 2)(2z + -6) && \text{(DL)}
\end{aligned}$$

25. First take the product of the outside terms, $3\gamma^2 \cdot 1 = 3\gamma^2$ Then we look at all the way we can factor this coefficent, $3 \mapsto$ as: $\pm[1, 3]$ We scan for a suitable pair to split 4γ into. Idenfiying $4\gamma = \gamma + 3\gamma$ as a suitable candidate. Then we stop talking start the doing!

$$3\gamma^2 + 4\gamma + 1 \quad \text{(given)}$$

$$=3\gamma^2 + (\gamma + 3\gamma) + 1 \quad \text{(Bi, The Split!)}$$

$$= \left(3\gamma^2 + \gamma\right) + (3\gamma + 1) \quad \text{(ALA)}$$

$$= (3\gamma + 1)\,\gamma + (3\gamma + 1)\,1 \quad \text{(DL, BI)}$$

$$= (3\gamma + 1)\,(\gamma + 1) \quad \text{(DL)}$$

26. First take the product of the outside terms, $6\theta^6 \cdot 45q^4 = 270\theta^6 q^4$ Then we look at all the way we can factor this coefficent, $270 \mapsto$ as: $\pm[1, 270]$, $\pm[2, 135]$, $\pm[3, 90]$, $\pm[5, 54]$, $\pm[6, 45]$, $\pm[9, 30]$, $\pm[10, 27]$, $\pm[15, 18]$ We scan for a suitable pair to split $33\theta^3 q^2$ into. Idenfiying $33\theta^3 q^2 = 18\theta^3 q^2 + 15\theta^3 q^2$ as a suitable candidate. Then we stop talking start the doing!

$$6\theta^6 + 33\theta^3 q^2 + 45q^4 \quad \text{(given)}$$

$$=6\theta^6 + \left(18\theta^3 q^2 + 15\theta^3 q^2\right) + 45q^4 \quad \text{(Bi, The Split!)}$$

$$= \left(6\theta^6 + 18\theta^3 q^2\right) + \left(15\theta^3 q^2 + 45q^4\right) \quad \text{(ALA)}$$

$$= \left(3\theta^3 + 9q^2\right) 2\theta^3 + \left(3\theta^3 + 9q^2\right) 5q^2 \quad \text{(DL, BI)}$$

$$= \left(3\theta^3 + 9q^2\right) \left(2\theta^3 + 5q^2\right) \quad \text{(DL)}$$

27. First take the product of the outside terms, $4x^2 \cdot 44 = 176x^2$ Then we look at all the way we can factor this coefficent, $176 \mapsto$ as: $\pm[1, 176]$, $\pm[2, 88]$, $\pm[4, 44]$, $\pm[8, 22]$, $\pm[11, 16]$ We scan for a suitable pair to split $-30x$ into. Idenfiying $-30x = -8x + -22x$ as a suitable candidate. Then we stop talking start the doing!

$$4x^2 + -30x + 44 \quad \text{(given)}$$

$$=4x^2 + (-8x + -22x) + 44 \quad \text{(Bi, The Split!)}$$

$$= \left(4x^2 + -8x\right) + (-22x + 44) \quad \text{(ALA)}$$

$$= (2x + -4)\,2x + (2x + -4) - 11 \quad \text{(DL, BI)}$$

$$= (2x + -4)\,(2x + -11) \quad \text{(DL)}$$

28. First take the product of the outside terms, $-r^2 \cdot -18 = 18r^2$ Then we look at all the way we can factor this coefficent, $18 \mapsto$ as: $\pm[1, 18]$, $\pm[2, 9]$, $\pm[3, 6]$ We scan for a suitable pair to split $-9r$ into. Idenfiying $-9r = -3r + -6r$ as a suitable candidate. Then we stop talking start the doing!

$$- r^2 + -9r + -18 \quad \text{(given)}$$

$$= - r^2 + (-3r + -6r) + -18 \quad \text{(Bi, The Split!)}$$

$$= \left(-r^2 + -3r\right) + (-6r + -18) \quad \text{(ALA)}$$

$$= (-r + -3)\,r + (-r + -3)\,6 \quad \text{(DL, BI)}$$

$$= (-r + -3)\,(r + 6) \quad \text{(DL)}$$

29. First take the product of the outside terms, $3\mu^2 \cdot 2 = 6\mu^2$ Then we look at all the way we can factor this coefficent, $6 \mapsto$ as: $\pm[1, 6]$, $\pm[2, 3]$ We scan for a suitable pair to split 5μ into. Idenfiying $5\mu = 3\mu + 2\mu$ as a suitable candidate. Then we stop talking start the doing!

$$3\mu^2 + 5\mu + 2 \quad \text{(given)}$$

$$=3\mu^2 + (3\mu + 2\mu) + 2 \quad \text{(Bi, The Split!)}$$

$$= \left(3\mu^2 + 3\mu\right) + (2\mu + 2) \quad \text{(ALA)}$$

$$= (\mu + 1)\,3\mu + (\mu + 1)\,2 \quad \text{(DL, BI)}$$

$$= (\mu + 1)\,(3\mu + 2) \quad \text{(DL)}$$

30. First take the product of the outside terms, $-2v^2 \cdot -27 = 54v^2$ Then we look at all the way we can factor this coefficent, $54 \mapsto$ as: $\pm[1, 54]$, $\pm[2, 27]$, $\pm[3, 18]$, $\pm[6, 9]$ We scan for a suitable pair to split $15v$ into. Idenfiying $15v = 9v + 6v$ as a suitable candidate. Then we stop talking start the doing!

$$- 2v^2 + 15v + -27 \quad \text{(given)}$$

$$= - 2v^2 + (9v + 6v) + -27 \quad \text{(Bi, The Split!)}$$

$$= \left(-2v^2 + 9v\right) + (6v + -27) \quad \text{(ALA)}$$

$$= (-2v + 9)\,v + (-2v + 9) - 3 \quad \text{(DL, BI)}$$

$$= (-2v + 9)\,(v + -3) \quad \text{(DL)}$$

Section 4.9

1.

$\square^3 + \triangle^3 = (\square + \triangle)\left(\square^2 - \square\triangle + \triangle^2\right)$

(..recall the SC Pattern)

$$
\begin{aligned}
&27v^3 + -216 && \text{(given)}\\
&= (3v)^3 + (-6)^3 && \text{(Bi, see SC pattern)}\\
&= (3v + -6)\left[(3v)^2 - (3v)(-6) + (-6)^2\right] && \text{(SC)}\\
&= (3v + -6)\left[9v^2 + 18v + 36\right] && \text{(BI)}
\end{aligned}
$$

2.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$

(..recall the pp2 Pattern)

$$
\begin{aligned}
&9w^2 + \frac{21w}{2} + \frac{49}{16} && \text{(given)}\\
&= (3w)^2 + 2\cdot(3w)\left(\frac{7}{4}\right) + \left(\frac{7}{4}\right)^2 && \text{(Bi, identified PP2 pattern)}\\
&= \left(3w + \frac{7}{4}\right)^2 && \text{(PP2)}
\end{aligned}
$$

3.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$

(..recall the pp2 Pattern)

$$
\begin{aligned}
&4t^2 + 4t + 1 && \text{(given)}\\
&= (2t)^2 + 2\cdot(2t)(1) + (1)^2 && \text{(Bi, identified PP2 pattern)}\\
&= (2t + 1)^2 && \text{(PP2)}
\end{aligned}
$$

4.

$(\square + \triangle)^3$

$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$

(..recall the pp3 Pattern)

$$
\begin{aligned}
&y^3 + -30y^2 + +300y + -1000 && \text{(given)}\\
&= (y)^3 + 3\cdot(y)^2(-10) + 3\cdot(y)(-10)^2 + (-10)^2 && \text{(Bi, seePP3 pattern)}\\
&= (y + -10)^3 && \text{(PP3)}
\end{aligned}
$$

5.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$

(..recall the pp2 Pattern)

$$
\begin{aligned}
&4z^2 + 16z + 16 && \text{(given)}\\
&= (2z)^2 + 2\cdot(2z)(4) + (4)^2 && \text{(Bi, identified PP2 pattern)}\\
&= (2z + 4)^2 && \text{(PP2)}
\end{aligned}
$$

6.

$(\square + \triangle)^3$

$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$

(..recall the pp3 Pattern)

$$
\begin{aligned}
&\theta^3 + 6\theta^2 + +12\theta + 8 && \text{(given)}\\
&= (\theta)^3 + 3\cdot(\theta)^2(2) + 3\cdot(\theta)(2)^2 + (2)^2 && \text{(Bi, seePP3 pattern)}\\
&= (\theta + 2)^3 && \text{(PP3)}
\end{aligned}
$$

7.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$

(..recall the pp2 Pattern)

$$
\begin{aligned}
&9V^2 + 54V + 81 && \text{(given)}\\
&= (3V)^2 + 2\cdot(3V)(9) + (9)^2 && \text{(Bi, identified PP2 pattern)}\\
&= (3V + 9)^2 && \text{(PP2)}
\end{aligned}
$$

8.

$\square^2 - \triangle^2 = (\square - \triangle)(\square + \triangle)$

(..recall the DS Pattern)

$$
\begin{aligned}
&4v^2 + -25 && \text{(given)}\\
&= (2v)^2 - (5)^2 && \text{(Bi, see DS pattern)}\\
&= (2v + 5)(2v - 5) && \text{(DS)}
\end{aligned}
$$

9.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$

(..recall the pp2 Pattern)

$$
\begin{aligned}
&x^2 + 2x + 1 && \text{(given)}\\
&= (x)^2 + 2\cdot(x)(1) + (1)^2 && \text{(Bi, identified PP2 pattern)}\\
&= (x + 1)^2 && \text{(PP2)}
\end{aligned}
$$

10.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$

(..recall the pp2 Pattern)

$$
\begin{aligned}
&4\mu^2 + \frac{4\mu}{7} + \frac{1}{49} && \text{(given)}\\
&= (2\mu)^2 + 2\cdot(2\mu)\left(\frac{1}{7}\right) + \left(\frac{1}{7}\right)^2 && \text{(Bi, identified PP2 pattern)}\\
&= \left(2\mu + \frac{1}{7}\right)^2 && \text{(PP2)}
\end{aligned}
$$

11.

$(\square + \triangle)^3$
$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$
(..recall the pp3 Pattern)

$27\lambda^3 + -81\lambda^2 + +81\lambda + -27$ (given)
$= (3\lambda)^3 + 3\cdot(3\lambda)^2(-3) + 3\cdot(3\lambda)(-3)^2 + (-3)^2$ (Bi, seePP3 pattern)
$= (3\lambda + -3)^3$ (PP3)

12.

$\square^2 - \triangle^2 = (\square - \triangle)(\square + \triangle)$
(..recall the DS Pattern)

$x^2 + -81$ (given)
$= (x)^2 - (9)^2$ (Bi, see DS pattern)
$= (x + 9)(x - 9)$ (DS)

13.

$(\square + \triangle)^3$
$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$
(..recall the pp3 Pattern)

$8q^3 + 72q^2 + +216q + 216$ (given)
$= (2q)^3 + 3\cdot(2q)^2(6) + 3\cdot(2q)(6)^2 + (6)^2$ (Bi, seePP3 pattern)
$= (2q + 6)^3$ (PP3)

14.

$\square^2 - \triangle^2 = (\square - \triangle)(\square + \triangle)$
(..recall the DS Pattern)

$v^2 + -4$ (given)
$= (v)^2 - (2)^2$ (Bi, see DS pattern)
$= (v + 2)(v - 2)$ (DS)

15.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$
(..recall the pp2 Pattern)

$9r^2 + 42r + 49$ (given)
$= (3r)^2 + 2\cdot(3r)(7) + (7)^2$ (Bi, identified PP2 pattern)
$= (3r + 7)^2$ (PP2)

16.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$
(..recall the pp2 Pattern)

$X^2 + 10X + 25$ (given)
$= (X)^2 + 2\cdot(X)(5) + (5)^2$ (Bi, identified PP2 pattern)
$= (X + 5)^2$ (PP2)

17.

$(\square + \triangle)^3$
$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$
(..recall the pp3 Pattern)

$27w^3 + -135w^2 + +225w + -125$ (given)
$= (3w)^3 + 3\cdot(3w)^2(-5) + 3\cdot(3w)(-5)^2 + (-5)^2$ (Bi, seePP3 pattern)
$= (3w + -5)^3$ (PP3)

18.

$(\square + \triangle)^3$
$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$
(..recall the pp3 Pattern)

$8\omega^3 + 108\omega^2 + +486\omega + 729$ (given)
$= (2\omega)^3 + 3\cdot(2\omega)^2(9) + 3\cdot(2\omega)(9)^2 + (9)^2$ (Bi, seePP3 pattern)
$= (2\omega + 9)^3$ (PP3)

19.

$(\square + \triangle)^3$
$= \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$
(..recall the pp3 Pattern)

$27\mu^3 + 216\mu^2 + +576\mu + 512$ (given)
$= (3\mu)^3 + 3\cdot(3\mu)^2(8) + 3\cdot(3\mu)(8)^2 + (8)^2$ (Bi, seePP3 pattern)
$= (3\mu + 8)^3$ (PP3)

20.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$
(..recall the pp2 Pattern)

$9r^2 + 36r + 36$ (given)
$= (3r)^2 + 2\cdot(3r)(6) + (6)^2$ (Bi, identified PP2 pattern)
$= (3r + 6)^2$ (PP2)

21.

$\square^3 + \triangle^3 = (\square + \triangle)\left(\square^2 - \square\triangle + \triangle^2\right)$
(..recall the SC Pattern)

$X^3 + 1000$ (given)
$= (X)^3 + (10)^3$ (Bi, see SC pattern)
$= (X + 10)\left[(X)^2 - (X)(10) + (10)^2\right]$ (SC)
$= (X + 10)\left[X^2 - 10X + 100\right]$ (BI)

22.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$ (..recall the pp2 Pattern)

$$V^2 + 4V + 4 \quad \text{(given)}$$
$$= (V)^2 + 2 \cdot (V)(2) + (2)^2 \quad \text{(Bi, identified PP2 pattern)}$$
$$= (V + 2)^2 \quad \text{(PP2)}$$

23.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$ (..recall the pp2 Pattern)

$$9Y^2 + 60Y + 100 \quad \text{(given)}$$
$$= (3Y)^2 + 2 \cdot (3Y)(10) + (10)^2 \quad \text{(Bi, identified PP2 pattern)}$$
$$= (3Y + 10)^2 \quad \text{(PP2)}$$

24.

$(\square + \triangle)^3 = \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$ (..recall the pp3 Pattern)

$$\xi^3 + 30\xi^2 + +300\xi + 1000 \quad \text{(given)}$$
$$= (\xi)^3 + 3 \cdot (\xi)^2 (10) + 3 \cdot (\xi)(10)^2 + (10)^2 \quad \text{(Bi, seePP3 pattern)}$$
$$= (\xi + 10)^3 \quad \text{(PP3)}$$

25.

$(\square + \triangle)^3 = \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$ (..recall the pp3 Pattern)

$$27\rho^3 + -108\rho^2 + +144\rho + -64 \quad \text{(given)}$$
$$= (3\rho)^3 + 3 \cdot (3\rho)^2 (-4) + 3 \cdot (3\rho)(-4)^2 + (-4)^2 \quad \text{(Bi, seePP3 pattern)}$$
$$= (3\rho + -4)^3 \quad \text{(PP3)}$$

26.

$\square^3 + \triangle^3 = (\square + \triangle)\left(\square^2 - \square\triangle + \triangle^2\right)$ (..recall the SC Pattern)

$$27\rho^3 + -512 \quad \text{(given)}$$
$$= (3\rho)^3 + (-8)^3 \quad \text{(Bi, see SC pattern)}$$
$$= (3\rho + -8)\left[(3\rho)^2 - (3\rho)(-8) + (-8)^2\right] \quad \text{(SC)}$$
$$= (3\rho + -8)\left[9\rho^2 + 24\rho + 64\right] \quad \text{(BI)}$$

27.

$(\square + \triangle)^3 = \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$ (..recall the pp3 Pattern)

$$R^3 + 12R^2 + +48R + 64 \quad \text{(given)}$$
$$= (R)^3 + 3 \cdot (R)^2 (4) + 3 \cdot (R)(4)^2 + (4)^2 \quad \text{(Bi, seePP3 pattern)}$$
$$= (R + 4)^3 \quad \text{(PP3)}$$

28.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$ (..recall the pp2 Pattern)

$$9u^2 + 60u + 100 \quad \text{(given)}$$
$$= (3u)^2 + 2 \cdot (3u)(10) + (10)^2 \quad \text{(Bi, identified PP2 pattern)}$$
$$= (3u + 10)^2 \quad \text{(PP2)}$$

29.

$(\square + \triangle)^3 = \square^3 + 3\square^2\triangle + 3\square\triangle^2 + \triangle^3$ (..recall the pp3 Pattern)

$$27W^3 + -54W^2 + +36W + -8 \quad \text{(given)}$$
$$= (3W)^3 + 3 \cdot (3W)^2 (-2) + 3 \cdot (3W)(-2)^2 + (-2)^2 \quad \text{(Bi, seePP3 pattern)}$$
$$= (3W + -2)^3 \quad \text{(PP3)}$$

30.

$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$ (..recall the pp2 Pattern)

$$4W^2 + \frac{16W}{9} + \frac{16}{81} \quad \text{(given)}$$
$$= (2W)^2 + 2 \cdot (2W)\left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 \quad \text{(Bi, identified PP2 pattern)}$$
$$= \left(2W + \frac{4}{9}\right)^2 \quad \text{(PP2)}$$

31.

$\square^2 - \triangle^2 = (\square - \triangle)(\square + \triangle)$ (..recall the DS Pattern)

$$9x^2 + -16 \quad \text{(given)}$$
$$= (3x)^2 - (4)^2 \quad \text{(Bi, see DS pattern)}$$
$$= (3x + 4)(3x - 4) \quad \text{(DS)}$$

32.

$$(\square + \triangle)^2 = \square^2 + 2\square\triangle + \triangle^2$$
(..recall the pp2 Pattern)

$$9q^2 + \frac{14q}{3} + \frac{49}{81} \quad \text{(given)}$$
$$= (3q)^2 + 2 \cdot (3q)\left(\frac{7}{9}\right) + \left(\frac{7}{9}\right)^2 \quad \text{(Bi, identified PP2 pattern)}$$
$$= \left(3q + \frac{7}{9}\right)^2 \quad \text{(PP2)}$$

33.

$$\square^2 - \triangle^2 = (\square - \triangle)(\square + \triangle)$$
(..recall the DS Pattern)

$$9t^2 + -9 \quad \text{(given)}$$
$$= (3t)^2 - (3)^2 \quad \text{(Bi, see DS pattern)}$$
$$= (3t + 3)(3t - 3) \quad \text{(DS)}$$

34.

$$\square^3 + \triangle^3 = (\square + \triangle)\left(\square^2 - \square\triangle + \triangle^2\right)$$
(..recall the SC Pattern)

$$w^3 + -216 \quad \text{(given)}$$
$$= (w)^3 + (-6)^3 \quad \text{(Bi, see SC pattern)}$$
$$= (w + -6)\left[(w)^2 - (w)(-6) + (-6)^2\right] \quad \text{(SC)}$$
$$= (w + -6)\left[w^2 + 6w + 36\right] \quad \text{(BI)}$$

35.

$$\square^2 - \triangle^2 = (\square - \triangle)(\square + \triangle)$$
(..recall the DS Pattern)

$$4r^2 + -64 \quad \text{(given)}$$
$$= (2r)^2 - (8)^2 \quad \text{(Bi, see DS pattern)}$$
$$= (2r + 8)(2r - 8) \quad \text{(DS)}$$

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