Famous IDs: Def. Identities

Main Idea

Trigonometric identities are a significant and essential portion of any trigonometry introductory course. In the first couple sections, we have introduced what 'identity' means, and we have introduced some of the most common and useful ways to establish wether a proposed equation is or is not an identity, namely, working on each side, looking at the graphs, tweaking an known identity, or something brilliant and creative. Moreover, the last section should have provided ample practice in proving identities. In fact, some of these will translate, almost verbatim, into very famous trig identities.

For the remainder of this chapter, we will turn our attention exclusively to trigonometric identities. Some of these trig identities are incredibly famous [and useful] and others are incredibly not famous. We will prove both types, the famous ones because they are famous, and the non-famous ones just for practice.

To that end, the time is here to introduce some of the most famous trigonometric identities.

(see http://www.mathhands.com/104/free/ids.pdf)

First Paragraph: The Definition Identities

Definition Identities

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$
$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

It should be noted that the following identities require no proving, since they follow immediately from the definitions of each of the functions.

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp}$$
$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{hyp}$$

EXAMPLE (working on one side) Prove the following is an identity.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Solution: Since every angle has a reference angle, the angle, θ, has a reference triangle. Let us label it as usual with opp, hyp, and adj sides, as appropriate. Then...

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$
\begin{aligned}
& \frac{\sin\theta}{\cos\theta} && \text{(right side)}\\
&= \frac{\frac{opp}{hyp}}{\frac{adj}{hyp}} && \text{(use definitions)}\\
&= \frac{\frac{opp}{hyp}}{\frac{adj}{hyp}} \cdot \frac{hyp}{hyp} && \text{(algebra)}\\
&= \frac{opp}{ajd} && \text{(algebra)}\\
&= \tan\theta && \text{(def.)}
\end{aligned}
$$

Therefore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The rest of the identities are proven in a similar manner and are left as important exercises for the student. With that, we now turn our attention to the second paragraph on the famous identity sheet.

Second Paragraph: The Co-Function Identities

Co-Function Identities

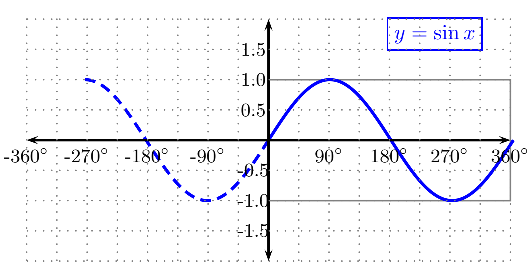
$$
\begin{aligned}
\sin\theta &= \cos(90^\circ - \theta) & \tan\theta &= \cot(90^\circ - \theta)\\
\sec\theta &= \csc(90^\circ - \theta) & \sin\theta &= \cos(\theta - 90^\circ)\\
-\cos\theta &= \sin(\theta - 90^\circ) & \cos\theta &= \sin(90^\circ - \theta)
\end{aligned}
$$

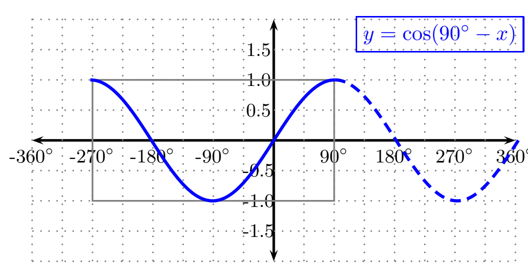
EXAMPLE: (look at graphs) Prove the following is an identity.

$$\sin x = \cos(90^\circ - x)$$

solution:

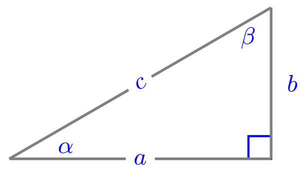
We will look at each of the graphs and compare to see if these are convincingly equal(Note to graph $\cos(90^\circ - x)$, we could use the *prep, scale, shift* method, thus we will graph the equivalent [prepared] version $\cos(-(x - 90^\circ))$... ie flipped then shifted right 90°).



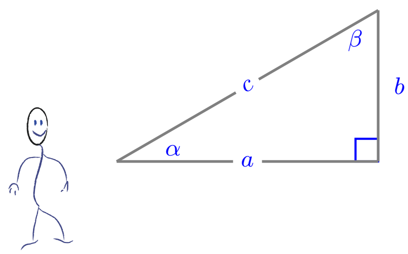


Some problems are so nice that they generate multiply solutions, in multiple ways. Here, we can't help but present a second solution to the exact same question. We will next prove the identity $\sin x = \cos(90^\circ - x)$ not by looking at the respective graphs but by something a little more interesting.

Suppose the angle x is between 0 and 90°, then the general reference triangle looks as such:



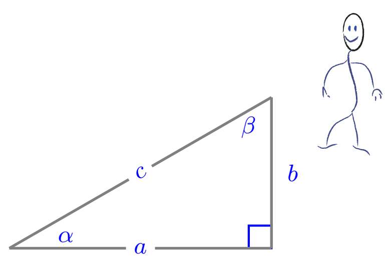
The key is to look at this triangle from different points of view. From the angle α point of view, the opposite side is b and the hypothenuse is c. Thus,



Thus,

$$\sin\alpha = \frac{b}{c}$$

On the other hand, if from the **Co**mplimentary angle, β, the adjacent side is b.



Thus,

$$\cos\beta = \frac{b}{c}$$

Therefore,

$$\sin\alpha = \cos\beta$$

moreover, α and β are complimentary, thus $\alpha + \beta = 90°$ OR $\beta = 90° - \alpha$

Therefore:

$$\sin\alpha = \cos(90° - \alpha)$$

The above argument assume α is between o and 90°, the reader is invited to generalize the argument for negative angles or for angles beyond the first quadrant, in fact the identity $\sin\alpha = \cos(90° - \alpha)$ holds true for any real angle, α.

Famous IDs: Def. Identities

1. [very famous] Determine if the following is an identity or not: Explain

$$\cot x = \frac{\cos x}{\sin x}$$

Solution: Yes, it is an identity, can be proven by working on the right hand side..

$$\begin{aligned}
\cot x &\stackrel{?}{=} \frac{\cos x}{\sin x} && (?) \\
&\stackrel{?}{=} \frac{\frac{adj}{hyp}}{\frac{opp}{hyp}} && \text{(def of sine, cosine, for ref triangle for angle } x) \\
&\stackrel{?}{=} \frac{adj}{opp} && \text{(algebra)} \\
&= \cot x && \text{(def of cot, DONE)}
\end{aligned}$$

2. [very famous] Determine if the following is an identity or not: Explain

$$\sec x = \frac{1}{\cos x}$$

Solution: Yes, it is an identity, can be proven by working on the right hand side..

$$\begin{aligned}
\sec x &\stackrel{?}{=} \frac{1}{\cos x} && (?) \\
&\stackrel{?}{=} \frac{1}{\frac{adj}{hyp}} && \text{(def of cosine, for ref triangle for angle } x) \\
&\stackrel{?}{=} \frac{hyp}{adj} && \text{(algebra)} \\
&= \sec x && \text{(def of secant, DONE)}
\end{aligned}$$

3. [very famous] Determine if the following is an identity or not: Explain

$$\csc x = \frac{1}{\sin x}$$

Solution: Yes, it is an identity, can be proven by working on the right hand side..

$$\csc x \overset{?}{=} \frac{1}{\sin x} \qquad (?)$$

$$\|$$

$$\overset{?}{=} \frac{1}{\frac{opp}{hyp}} \qquad \text{(def of cosine, for ref triangle for angle } x\text{)}$$

$$\|$$

$$\overset{?}{=} \frac{hyp}{opp} \qquad \text{(algebra)}$$

$$\|$$

$$= \csc x \qquad \text{(def of cosecant, DONE)}$$

4. [very famous] Determine if the following is an identity or not: Explain

$$\sin(2x) = 2\sin x$$

Solution: not an identity, can be seen by comparing the graphs OR can be seen by checking a few values, $x = 90°$ for example.

5. [very famous] Determine if the following is an identity or not: Explain

$$\sin x = \frac{1}{\csc x}$$

6. [very famous] Determine if the following is an identity or not: Explain

$$\cos x = \frac{1}{\sec x}$$

7. [little famous] Determine if the following is an identity or not: Explain

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$$

Solution: tweak a known identity;

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{(known identity)}$$

$$(\tan x)^2 = \left(\frac{\sin x}{\cos x}\right)^2 \qquad \text{(Square both sides)}$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} \qquad \text{(algebra)}$$

8. [little famous] Determine if the following is an identity or not: Explain

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x}$$

9. [little famous] Determine if the following is an identity or not: Explain

$$\sec^2 x = \frac{1}{\cos^2 x}$$

10. [very famous] Determine if the following is an identity or not: Explain

$$\tan\theta = \cot(90^\circ - \theta)$$

Solution: look at graphs

11. [very famous] Determine if the following is an identity or not: Explain

$$\sec\theta = \csc(90^\circ - \theta)$$

Solution: look at graphs OR tweak a known identity

$$\cos\theta = \sin(90^\circ - \theta) \qquad \text{(known \& proven (or can prove by graphs))}$$

$$\frac{1}{\cos\theta} = \frac{1}{\sin(90^\circ - \theta)} \qquad \text{(algebra)}$$

$$\sec\theta = \csc(90^\circ - \theta) \qquad \text{(def identities)}$$

12. [very famous] Determine if the following is an identity or not: Explain

$$\sin\theta = \cos(\theta - 90^\circ)$$

Solution: look at graphs

13. [very famous] Determine if the following is an identity or not: Explain

$$-\cos\theta = \sin(\theta - 90^\circ)$$

Solution: look at graphs

14. [very famous] Determine if the following is an identity or not: Explain

$$\cos\theta = \sin(90^\circ - \theta)$$