Famous IDs: Even/Odd Identities

Main Idea

We continue to work our way through the most famous trig identities (see http://www.mathhands.com/104/free/ids.pdf), working top to bottom on the left column. The student is advised to print this sheet and systematically check off each identity as she/he understands it, proves it, and makes it her/his own. Moreover, we will often use the *tweak a known identity* method to prove some of these. When doing so, it is important to *only* use identities that have already been proven, and *not* to use identities form the future [logical future]. Said differently, we work our way down the sheet, to prove any one identify, we can use any other identity which appears before it on the sheet, and nothing that appears ahead of it, that is assuming we go in order. This will help make the proofs logically sound, and not flawed with circular arguments.

That said, we continue with the next paragraph, the paragraph containing the identities which describe which functions are even and which are odd. But first, a little word about general even and odd functions.

Definition: Even Functions

Consider all functions with domain and codomain, $\mathbb{R}$. Some of these have a very interesting property. Namely, they make no distinction between negative and positive numbers. For example, consider

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = x^2$$

The domain for this function is the real numbers, and the codomain is also the real numbers. Now consider how f treats positive and negative numbers. For example, how does f treat 3 as compared with -3?

$$f(3) = (3)^2 = 9 \qquad \text{while} \qquad f(-3) = (-3)^2 = 9$$

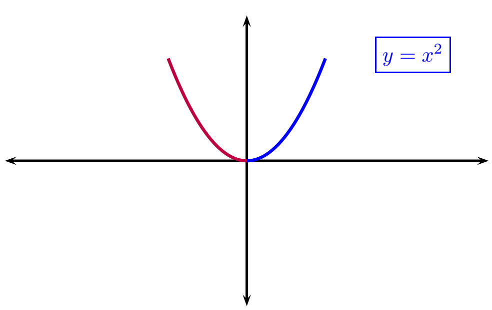
Indeed, f maps both 3 and -3 to the same number in the co-domain, 9. In this sense, it makes no distinction between 3 and -3. More generally, it will make no distinction between 5 and -5, 7 and -7, etc.. In fact, we can prove it make no distinction between any real number x and $-x$, since:

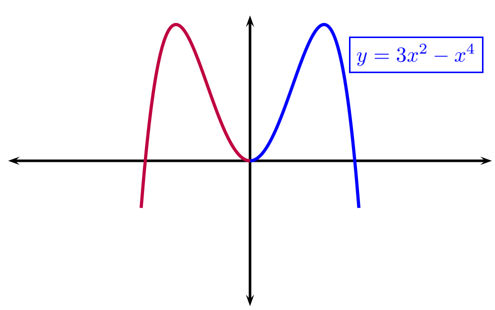
$$f(x) = (x)^2 = x^2 \qquad \text{while} \qquad f(-x) = (-x)^2 = x^2$$

This proves that for all $x \in \mathbb{R}$,

$$f(x) = f(-x)$$

Function that have this very special property are call *even functions*. Moreover, the graphs of even functions have the distinction of symmetry about the y axis. That is to say, if one 'folded' the graph along the y axis the part on the right side of the graph would math the part on the left. Here are some examples of even functions and their graphs.





To say that $f(-x) = f(x)$ for all x in domain of f, is equivalent to saying that a point, (x, y), is on the graph of f if and only if $(-x, y)$ is also on the graph, which is also equivalent to saying that the graph is symmetric about the y axis.

Definition: Odd Functions

While even functions, by definition, map every x and $-x$ to the same number, odd function are defined to be those functions that map $-x$ to the opposite of where x gets mapped to. Said differently, if

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}$$

Then we call such function *odd*. Another characterization of odd functions is that if x goes to y under such function then, $-x$ goes to $-y$. Here is a classic example, consider

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = x^3$$

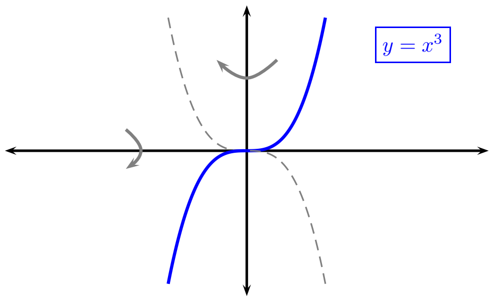
The domain for this function is the real numbers, and the codomain is also the real numbers. Now consider how f treats positive and negative numbers. For example, how does f treat 5 as compared with -5?

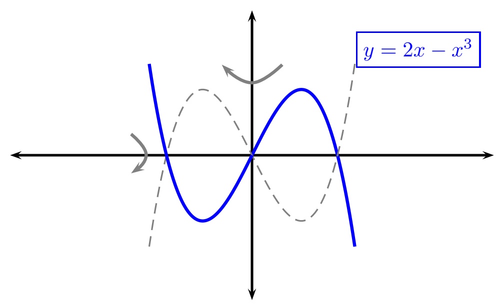
$$f(5) = (5)^3 = 125 \qquad \text{while} \qquad f(-5) = (-5)^5 = -125$$

Yet another way to put it is that one can always 'factor' out the negative *thru* the function as in the above example,

$$f(-5) = -f(5)$$

On its graph, what that means,for an odd function, is that the graph is not changed when flipped about y-axis followed by a flip about the x axis. Here are a couple examples of classic odd functions along with their respective graphs.





Example: Prove a Function is Odd

Now, besides looking at the symmetry of the graph, one may proceed as usual, with one of the other 3 ways we've practiced for proving identities. For example, consider proving that the above function is odd by working on each side [the method].

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = 2x - x^3$$

To prove f is odd, we must prove

$$f(-x) = -f(x)$$

for all real values of x.

We proceed to work on the left side:

$$
\begin{aligned}
f(-x) & & \text{(given left side)}\\
2(-x) - (-x)^3 &= & \text{(sub into } f)\\
-2x + x^3 &= & \text{(algebra)}\\
-\left(2x - x^3\right) &= & \text{(algebra)}\\
-f(x) &= & \text{(sub } f(x))
\end{aligned}
$$

Thus, we have proven f is odd by proving

$$f(-x) = -f(x)$$

Finally, we focus on answering which, *if any*, of our famous trig functions are even, odd, or neither. Thus we have....

The Third Paragraph: The Even/Odd Identities

Even & Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

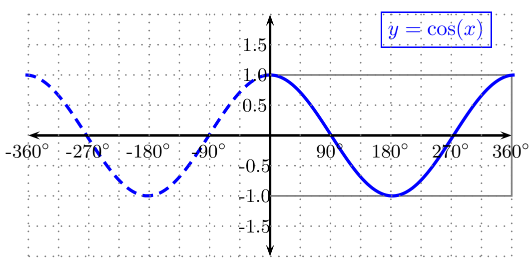
EXAMPLE: (look at graphs) Prove that cosine is even.

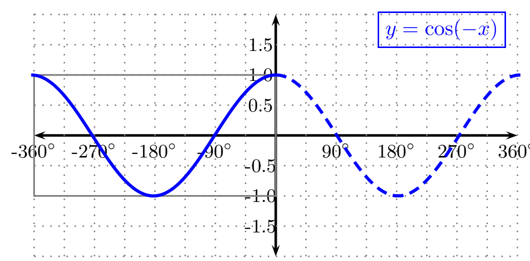
We must prove that for all possible real angles, x,

$$\cos(x) = \cos(-x)$$

solution:

We will look at each of the graphs and compare to see if these are convincingly equal.





Famous IDs: Even/Odd Identities

1. Prove $f(x) = \sin x$ is and odd function. i.e. prove
$$\sin(-x) = -\sin(x)$$

2. Prove $f(x) = \cos x$ is and even function. i.e. prove
$$\cos(-x) = \cos(x)$$

3. Prove $f(x) = \tan x$ is and odd function. i.e. prove
$$\tan(-x) = -\tan(x)$$

4. Prove $f(x) = \cot x$ is and odd function. i.e. prove
$$\cot(-x) = -\cot(x)$$

5. Prove $f(x) = \csc x$ is and odd function. i.e. prove
$$\csc(-x) = -\csc(x)$$

6. Prove $f(x) = \sec x$ is and even function. i.e. prove
$$\sec(-x) = \sec(x)$$

7. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = 2x - x^3 + x^5$$

8. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = 2x - x^5$$

9. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = 2x^2 - x^6$$

10. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = x^2 + 1$$

11. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = x^2 + x$$

12. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = x^2 + x^7$$

13. Determine if the following function is even, odd, or neither.
$$f : \mathbb{R} \to \mathbb{R} \qquad \text{with} \qquad f(x) = x^3 + 1$$