Famous IDs: Pythagoras Identities

Main Idea

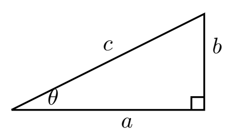
The previous section was extremely important for a few reasons. First, you should now be comfortable understanding what an identity is. Second, we saw methods used to establish identities, working on each side independently, tweaking a known identity, graphing the functions and comparing the graphs, and last, proving identities by starting with a very creative idea. Moreover, we introduced the first dose of fundamental and famous trigonometric identities, most or all of which were proved in the corresponding assignment.

Now, the task at hand is to expand the list of very famous trigonometric identities, and to practice our proving skills. Below is our next dose of famous identities, the first of which is *the* most famous of them all, the pythagoras identities.

Pythagoras Identities

$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$
$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$\tan^2\theta = \sec^2\theta - 1$	$\cot^2\theta + 1 = \csc^2\theta$
$\cot^2\theta = \csc^2\theta - 1$	$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$
$\tan\theta = \pm\sqrt{\sec^2\theta - 1}$	$\sec\theta = \pm\sqrt{\tan^2\theta + 1}$

EXAMPLE 1 (work on one side) The first identity in this family is the most famous trigonometric identity. We will prove it by working on one side. Without loss of generality, we can assume the following is a reference triangle for the angle θ



$\sin^2\theta + \cos^2\theta$	$\stackrel{?}{=}$	1	want to know
$\left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2$	|		definitions of sine and cosine
$\frac{b^2}{c^2} + \frac{a^2}{c^2}$	|		algebra clean up
$\frac{b^2+a^2}{c^2}$	|		algebra clean up
$\frac{c^2}{c^2}$	|		Pythagoras Theorem, see reference triangle
1	$=$	1	algebra clean up

EXAMPLE 2 (tweak a known identity) Prove the following is an identity.

$$\tan^2\theta + 1 = \sec^2\theta$$

Solution: We will prove this identity by tweaking the previous one, the Pythagoras Identity.

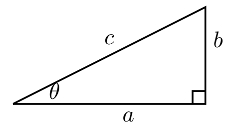
$$\begin{aligned}
\sin^2\theta + \cos^2 &= 1 && \text{We already know this one, proven above!} \\
\frac{1}{\cos^2\theta}\left(\sin^2\theta + \cos^2\right) &= \frac{1}{\cos^2\theta}(1) && \text{multiply both sides} \\
\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} && \text{algebra} \\
\left(\frac{\sin\theta}{\cos\theta}\right)^2 + \left(\frac{\cos\theta}{\cos\theta}\right)^2 &= \left(\frac{1}{\cos\theta}\right)^2 && \text{algebra} \\
(\tan\theta)^2 + (1)^2 &= (\sec\theta)^2 && \text{proven identity} \\
\tan^2\theta + 1 &= \sec^2\theta && \text{done!}
\end{aligned}$$

Famous IDs: Pythagoras Identities

1. Prove and OWN [VERY famous identity].

$$\sin^2\theta + \cos^2\theta = 1$$

Solution:



$\sin^2\theta + \cos^2\theta$	$\stackrel{?}{=}$	1	want to know
$\left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2$	|		definitions of sine and cosine
$\frac{b^2}{c^2} + \frac{a^2}{c^2}$	|		algebra clean up
$\frac{b^2+a^2}{c^2}$	|		algebra clean up
$\frac{c^2}{c^2}$	|		Pythagoras Theorem, see reference triangle
1	$=$	1	algebra clean up

2. Prove and OWN [VERY famous identity].

$$\tan^2\theta + 1 = \sec^2\theta$$

Solution: Solution: We will prove this identity by tweaking the previous one, the Pythagoras Identity.

$\sin^2\theta + \cos^2$	$=$	1	We already know this one, proven above!
$\frac{1}{\cos^2\theta}\left(\sin^2\theta + \cos^2\right)$	$=$	$\frac{1}{\cos^2\theta}(1)$	multiply both sides
$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}$	$=$	$\frac{1}{\cos^2\theta}$	algebra
$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + \left(\frac{\cos\theta}{\cos\theta}\right)^2$	$=$	$\left(\frac{1}{\cos\theta}\right)^2$	algebra
$(\tan\theta)^2 + (1)^2$	$=$	$(\sec\theta)^2$	proven identities
$\tan^2\theta + 1$	$=$	$\sec^2\theta$	done!

3. Prove and OWN [VERY famous identity].

$$\tan^2\theta = \sec^2\theta - 1$$

Solution: by tweaking

$$\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta && \text{(proven identity)} \\ \tan^2\theta &= \sec^2\theta - 1 && \text{(algebra)} \end{aligned}$$

4. Prove and OWN [VERY famous identity].

$$\cot^2\theta + 1 = \csc^2\theta$$

Solution: Solution: We will prove this identity by tweaking the previous one, the Pythagoras Identity.

$\sin^2\theta + \cos^2$	$=$	1	We already know this one, proven above!
$\frac{1}{\sin^2\theta}\left(\sin^2\theta + \cos^2\right)$	$=$	$\frac{1}{\sin^2\theta}(1)$	multiply both sides
$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$	$=$	$\frac{1}{\sin^2\theta}$	algebra
$\left(\frac{\sin\theta}{\sin\theta}\right)^2 + \left(\frac{\cos\theta}{\sin\theta}\right)^2$	$=$	$\left(\frac{1}{\sin\theta}\right)^2$	algebra
$(1)^2 + (\cot\theta)^2$	$=$	$(\csc\theta)^2$	proven identities
$1 + \cot^2\theta$	$=$	$\csc^2\theta$	done!

5. Prove and OWN [VERY famous identity].

$$\cot^2\theta = \csc^2\theta - 1$$

Solution: by tweaking

$$\begin{aligned} \cot^2\theta + 1 &= \csc^2\theta && \text{(proven identity)} \\ \cot^2\theta &= \csc^2\theta - 1 && \text{(algebra)} \end{aligned}$$

6. Prove the following famous identity

$$\sin^2\theta = 1 - \cos^2\theta$$

Solution: this is done by starting with a known (proven and famous) one $\sin^2\theta + \cos^2\theta = 1$. Then we isolate $\sin^2\theta$... then we are done!

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{(famous \& proven)}$$
$$= 1 - \cos^2\theta \quad \text{(algebra, sub } \cos^2\theta \text{ from both sides)}$$

7. Prove and OWN [VERY famous identity].

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

Solution: tweak....

$$\sin^2\theta = 1 - \cos^2\theta \quad \text{(proven)}$$
$$\sin\theta = \pm\sqrt{1-\cos^2\theta} \quad \text{(Square root property (algebra))}$$

8. Prove and OWN [VERY famous identity].

$$\tan\theta = \pm\sqrt{\sec^2\theta - 1}$$

9. Prove and OWN [VERY famous identity].

$$\sec\theta = \pm\sqrt{\tan^2\theta + 1}$$

10. Prove the following famous identity

$$\tan^2\theta = (\sec\theta - 1)(\sec\theta + 1)$$

Solution: this is done by starting with a known (proven and famous) one $\tan^2\theta = \sec^2\theta - 1$. Then we factor the right side. then we are done!

$$\tan^2\theta = \sec^2\theta - 1 \quad \text{(famous \& proven)}$$
$$= (\sec\theta - 1)(\sec\theta + 1) \quad \text{(algebra, factor diff of squares)}$$

11. Prove the following *is* an identity OR prove the following *is not* an identity

$$\frac{1}{1-\sin\theta} = \sec^2\theta + \tan\theta\sec\theta$$

Solution: Famous method.... work on one side... I would start working on right side.. Famous idea.. SOMETIMES.... it helps to turn everything into sines and cosines..

$$\frac{1}{1-\sin\theta} \stackrel{?}{=} \sec^2\theta + \tan\theta\sec\theta \quad = \quad \text{(abandon =)}$$

$$\frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta} \quad = \quad \text{(famous \& proven ids)}$$

$$\frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta} \quad = \quad \text{(algebra)}$$

$$\frac{1+\sin\theta}{\cos^2\theta} \quad = \quad \text{(algebra)}$$

$$\frac{1+\sin\theta}{1-\sin^2\theta} \quad = \quad \text{(famous \& proven ids)}$$

$$\frac{1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)} \quad = \quad \text{(algebra, factor)}$$

$$= \frac{1}{1-\sin\theta} \quad = \quad \text{(algebra, simplify, kachin, kachin, that is why they pay me)}$$

12. Prove the following *is* an identity OR prove the following *is not* an identity

$$\frac{1}{1-\sec\theta} = \cot^2\theta + \cot^2\theta\sec\theta$$

Solution: this is not an identity

13. Prove the following *is* an identity OR prove the following *is not* an identity

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$$

Solution: Famous method.... work on one side... I would start working on left side.. Famous idea.. SOMETIMES.... it helps to start on the side that looks more complex Another famous idea SOMETIMES.... its good to use conjugates, such as a-b and a+b

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} \stackrel{?}{=} \quad 2\sec^2\theta \qquad \text{(abandon =)}$$

$$\frac{1}{1-\sin\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta} + \frac{1}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta} \qquad \text{(conjugate idea, mult top and bottom)}$$

$$\frac{1+\sin\theta}{1-\sin^2\theta} + \frac{1-\sin\theta}{1-\sin^2\theta} \qquad \text{(conjugates always clean up nicely)}$$

$$\frac{1+\sin\theta}{\cos^2\theta} + \frac{1-\sin\theta}{\cos^2\theta} \qquad \text{(famous \& proven ids)}$$

$$\frac{2}{\cos^2\theta} \qquad \text{(combine into one fraction)}$$

$$2\sec^2\theta = \qquad \text{(famous \& proven ids)}$$

14. Prove the following *is* an identity OR prove the following *is not* an identity

$$\cos^4\theta = 1 - 2\sin^2 + \sin^4\theta$$

Solution: Famous method.... work on one side... I would start working on left side.. Famous idea.. SOMETIMES.... it helps to start on the side that has higher exponents...

$$\cos^4\theta \overset{?}{=} 1 - 2\sin^2 + \sin^4\theta \quad \text{(abandon =)}$$

$$\cos^2\theta\cos^2\theta \quad \text{(4th power is the same as square times squared, algebra)}$$

$$(1-\sin^2\theta)(1-\sin^2\theta) \quad \text{(famous \& proven ids)}$$

$$1 - 2\sin\theta + \sin^4\theta = \quad \text{(FOIL, from grades school algebra... kachin!)}$$

15. Prove the following *is* an identity OR prove the following *is not* an identity

$$\cos^6\theta = 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta$$

Solution: Famous idea.. its no fun if the teacher give you all the answers..... you try this one.. try hard... start on the left side.. $\cos^6\theta$ is $\cos^2\theta\cdot\cos^2\theta\cdot\cos^2\theta$ etc... etc..

16. Practice **Work on each side**: Assume there is an interesting world in which $x^2 + y^2 = 1$ for all values of x and y. In such world, Determine if the following is an identity, if so answer TRUE, if not answer FALSE.

$$2x^2 - 1 = 1 - 2y^2$$

A. True

B. False

Solution:

$$\begin{aligned}
&1 - 2y^2 && \text{(work on right side)}\\
&= x^2 + y^2 - 2y^2 && \text{(sub } 1{=}x^2 + y^2\text{)}\\
&= x^2 - y^2 && \text{(algebra)}\\
&= x^2 - y^2 + -x^2 + x^2 && \text{(clever! add zero)}\\
&= x^2 - (y^2 + x^2) + x^2 && \text{(factor negative)}\\
&= x^2 - (1) + x^2 && \text{(sub)}\\
&= 2x^2 - 1 && \text{(aglebra)}
\end{aligned}$$

ALTERNATIVELY, observe $x^2 + y^2 = 1$ implies $x^2 = 1 - y^2$, thus..

$$1 - 2y^2 \quad \text{(work on right side)}$$
$$= 1 - 2(1 - x^2) \quad \text{(the tweaked sub)}$$
$$= 2x^2 - 1 \quad \text{(aglebra)}$$

17. Practice **Work on each side**: Determine if the following is an identity, prove your answer

$$2\cos^2 x - 1 = 1 - 2\sin^2 x$$

A. True
B. False

Solution:

$$1 - 2\sin^2 x \quad \text{(work on right side)}$$
$$= \cos^2 x + \sin^2 x - 2\sin^2 x \quad \text{(sub } 1 = \cos^2 x + \sin^2 x\text{)}$$
$$= \cos^2 x - \sin^2 x \quad \text{(algebra)}$$
$$= \cos^2 x - \sin^2 x + -\cos^2 x + \cos^2 x \quad \text{(clever! add zero)}$$
$$= \cos^2 x - (\sin^2 x + \cos^2 x) + \cos^2 x \quad \text{(factor negative)}$$
$$= \cos^2 x - (1) + \cos^2 x \quad \text{(sub)}$$
$$= 2\cos^2 x - 1 \quad \text{(aglebra)}$$

ALTERNATIVELY, observe $\cos^2 x + \sin^2 x = 1$ implies $\cos^2 x = 1 - \sin^2 x$, thus..

$$1 - 2\sin^2 x \quad \text{(work on right side)}$$
$$= 1 - 2(1 - \cos^2 x) \quad \text{(the tweaked sub)}$$
$$= 2\cos^2 x - 1 \quad \text{(aglebra)}$$

18. Practice **Work on each side**: Assume there is an interesting world in which $x^2 + y^2 = 1$ for all values of x and y. In such world, Determine if the following is an identity, if so answer TRUE, if not answer FALSE.

$$(x + y)^2 = 2xy + 1$$

A. True
B. False

Solution: work on the left side again..

$$
\begin{aligned}
&(x+y)^2 && \text{(left side)}\\
&x^2+2xy+y^2 && \text{(Expand)}\\
&2xy+(x^2+y^2) && \text{(Commute, associate, prep to sub)}\\
&2xy+(1) && \text{(Commute, associate, prep to sub)}
\end{aligned}
$$

Therefore TRUE

19. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$(\cos x+\sin x)^2 = 2\cos x\sin x + 1$$

A. True

B. False

Solution: work on the left side again..

$$
\begin{aligned}
&(\cos x+\sin x)^2 && \text{(left side)}\\
&\cos^2 x+2\cos x\sin x+\sin^2 x && \text{(Expand)}\\
&2\cos x\sin x+(\cos^2 x+\sin^2 x) && \text{(Commute, associate, prep to sub)}\\
&2\cos x\sin x+(1) && \text{(Commute, associate, prep to sub)}
\end{aligned}
$$

Therefore TRUE

20. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\cos^4 x - \sin^4 x = 2\cos^4 x - 2\cos^2 x + 1$$

A. True

B. False

21. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$(\cos x+\sin x)^4 = -4\cos^4 x + 4\cos^2 x + 4\cos x\sin x + 1$$

A. True

B. False

22. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\cos x - \sin x = \frac{1}{\cos x + \sin x}$$

A. True

B. False

23. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x} = \frac{2\cos x}{1 - 2\sin^2 x}$$

A. True

B. False

24. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x} = \frac{2\cos x}{2\cos^2 x - 1}$$

A. True

B. False

25. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\frac{1}{1 - 2\sin^2 x} = \frac{1}{2\cos^2 x - 1}$$

A. True

B. False

26. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\cos x - \sin x = \frac{1}{2\cos x + \sin x}$$

A. True

B. False

27. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\frac{1}{\cos x + \sin x} - \frac{1}{\cos x - \sin x} = \frac{\cos x}{1 - 2\sin^2 x}$$

A. True

B. False

28. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x} = \frac{2\cos x}{1 - 2\cos^2 x}$$

A. True

B. False

29. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\cos^6 x + 3\cos^4 x \sin^2 x + 3\cos^2 x \sin^4 x + \sin^6 x + 2 = 3$$

A. True

B. False

30. Practice **Work on each side**: Determine if the following is an identity, prove your answer.

$$\cos^6 x + 3\cos^4 x \sin^2 x + 3\cos^2 x \sin^4 x + \sin^6 x - \cos^2 x = \sin^2 x$$

A. True

B. False

31. Prove the following *is* an identity OR prove the following *is not* an identity

$$\cos^6\theta = -2 + 3\cos^2\theta + 3\sin^4\theta - \sin^6\theta$$

32. Prove the following *is* an identity OR prove the following *is not* an identity

$$1 - \cos^4\theta = \sin^2\theta + \sin^2\theta\cos^2\theta$$

33. (**)Prove the identity

$$\sin(2x) = 2\sin x\cos x$$