Famous IDs: Sum-Angle Identities

Main Idea

We continue to expand the list of very famous trigonometric identities, and to practice our proving skills. We now prove the second most famous/most important trigonometric identity. By tweaking this identity, we will derive many, many more famous and important trigonometric identities. In some sense, this identity, $\cos(a-b) = \cos a \cos b + \sin a \sin b$, is the *Mother Of Them All*. We are now ready to learn *it* as well as all the famous variations listed below.

Sum-Angle Identities

the M.O.T.A. $\longmapsto$

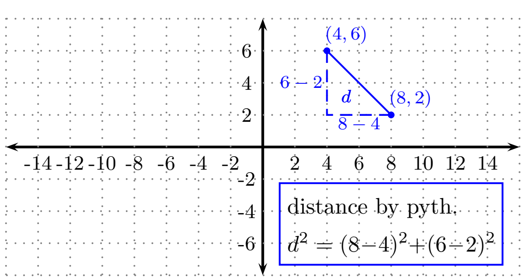
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

EXAMPLE 1 Mother Of Them All (by starting with something amazingly creative) Prove the following identity

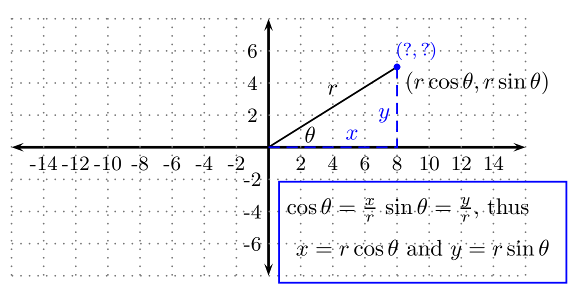
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Solution: In order to prove this one, we need several key ingredients. First, we need to be well versed in finding distances between two points. That is, we need to know that the distance squared, d^2, between two points, (A, B) and (C, K) is given by $d^2 = (A-C)^2 + (B-K)^2$. Next, we need to be well versed in finding the x and the y coordinates for a reference triangle where the hypothenuse is r and the angle is θ. Finally, we need an amazingly creative idea, apparently unrelated at first, but obviously divinely inspired once it delivers the goods.

preliminaries: review distance



review coordinates



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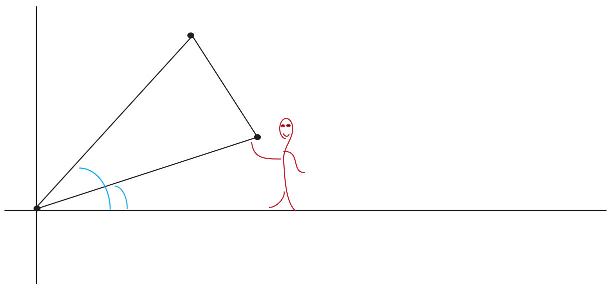
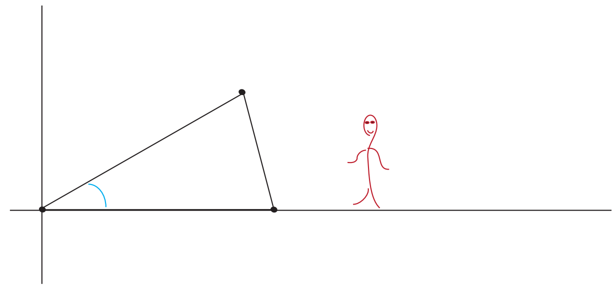
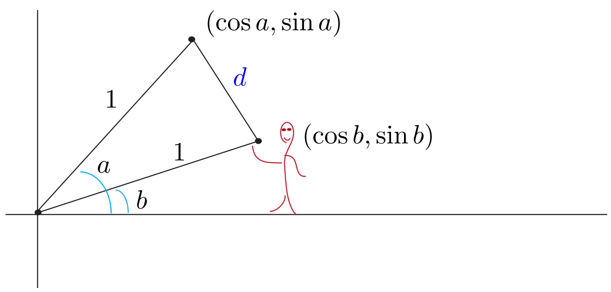
the brilliant idea: hold this triangle up, then walk away...

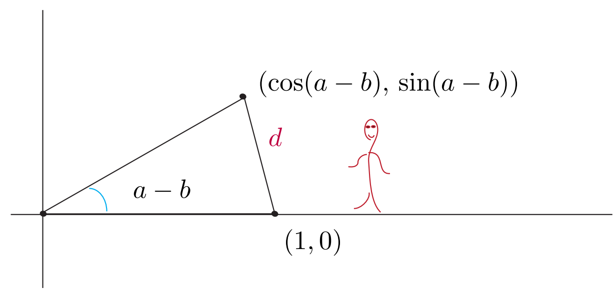




the brilliant idea: hold this triangle up, then walk away...

Now, the idea is to calculate the distance, d, in both cases, before the triangle falls and after it has fallen.





$d^2 = (\cos a - \cos b)^2 + (\sin a - \sin b)^2$ (from dist. formula)

$d^2 = \cos^2 a - 2\cos a\cos b + \cos^2 b + \sin^2 a - 2\sin a\sin b + \sin^2 b$ (FOIL, alg)

$d^2 = \cos^2 a + \sin^2 a + \cos^2 b + \sin^2 b - 2\cos a\cos b - 2\sin a\sin b$ (to use Pythagoras ID, alg)

$d^2 = 1 + 1 - 2\cos a\cos b - 2\sin a\sin b$ (use Pythagoras ID, alg)

$d^2 = 2 - 2\cos a\cos b - 2\sin a\sin b$ (alg)

$d^2 = [\cos(a-b) - 1]^2 + [\sin(a-b) - 0]^2$ (from dist. formula)

$d^2 = \cos^2(a-b) - 2\cos(a-b) + 1 + \sin^2(a-b)$ (FOIL, alg)

$d^2 = \cos^2(a-b) + \sin^2(a-b) - 2\cos(a-b) + 1$ (to use Pythagoras ID, alg)

$d^2 = 1 - 2\cos(a-b) + 1$ (use Pythagoras ID, alg)

$d^2 = 2 - 2\cos(a-b)$ (alg)

Now, we conclude the square of the distance, d^2 is the same before and after the triangle has fallen. Thus,...

$$d^2 = d^2 \qquad \text{(from above)}$$
$$2 - 2\cos a\cos b - 2\sin a\sin b = 2 - 2\cos(b-a) \qquad \text{(from work above)}$$
$$-2\cos a\cos b - 2\sin a\sin b = -2\cos(b-a) \qquad \text{(algebra)}$$
$$\cos a\cos b + \sin a\sin b = \cos(b-a) \qquad \text{(algebra)}$$

yip-kaei-yeah!!

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EXAMPLE 2 Famous Sum-Angle Id. (by tweaking a known Id.) Prove the following identity

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Solution: We begin with the MOTA Identity, which we have just proven. Since it is true for all angles a and b, we can apply it for $a = x$ and $b = -y$. After these values are substituted, we clean it up and voila!

$\cos(a-b)$	$=$	$\cos a \cos b + \sin a \sin b$	Known Id, MOTA
$\cos[x-(-y)]$	$=$	$\cos x \cos(-y) + \sin x \sin(-y)$	the Tweak, substitute $a = x$ and $b = -y$
$\cos(x+y)$	$=$	$\cos x \cos y + \sin x \sin(-y)$	clean up, cosine is an even function, see Famous Ids
$\cos(x+y)$	$=$	$\cos x \cos y + (\sin x)(-\sin y)$	clean up, sine is odd function, see Famous Ids
$\cos(x+y)$	$=$	$\cos x \cos y - \sin x \sin y$	clean up, QED

EXAMPLE 3 Sum-Angle for Sine (by tweaking a known Id.) Prove the following identity

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

Solution: we will tweak a known identity. Recall also the Co-Function identities. The main difference between a sine and a cosine is the 'co' for co-mplimentary angles, thus replacing any angle θ with $90^\circ - \theta$ results in the co-function ratio. With this in mind we will substitute $x = (90^\circ - a)$ and $y = b$

$\cos(x+y)$	$=$	$\cos x \cos y - \sin x \sin y$	known ID
$\cos[(90^\circ - a) + b]$	$=$	$\cos(90^\circ - a)\cos b - \sin(90^\circ - a)\sin b$	the tweak, sub
$\cos[90^\circ - (a-b)]$	$=$	$\cos(90^\circ - a)\cos b - \sin(90^\circ - a)\sin b$	algebra
$\sin[(a-b)]$	$=$	$\sin a \cos b - \cos a \sin b$	co-function Ids
$\sin(a-b)$	$=$	$\sin a \cos b - \cos a \sin b$	goods delivered! famous Id!

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Famous IDs: Sum-Angle Identities

1. Prove and OWN everyone of these famous identities.

Sum-Angle Identities

the M.O.T.A. $\longmapsto$

$$\begin{aligned} \cos(a-b) &= \cos a \cos b + \sin a \sin b \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b \\ \tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\ \tan(a-b) &= \frac{\tan a - \tan b}{1 + \tan a \tan b} \end{aligned}$$

2. Prove the following non-famous identity.

$$\cos(4x) = \cos x \cos 3x - \sin x \sin 3x$$

3. Prove the following non-famous identity.

$$\cos(3x) = \cos x \cos 2x - \sin x \sin 2x$$

4. Prove the following non-famous identity.

$$\cos(2x) = \cos x \cos x - \sin x \sin x$$

5. Prove the following non-famous identity.

$$\cos(4x) = \cos 2x \cos 2x - \sin 2x \sin 2x$$

6. Prove the following non-famous identity.

$$\cos(5x) = \cos 2x \cos 3x - \sin 2x \sin 3x$$

7. Prove the following non-famous identity.

$$\tan(10x) = \frac{\tan 12x - \tan 2x}{1 + \tan 12x \tan 2x}$$

8. Prove the following non-famous identity.

$$\tan(10x) = \frac{\tan 8x + \tan 2x}{1 - \tan 8x \tan 2x}$$

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9. Prove the following non-famous identity.

$$\cos(8x) = \cos 10x \cos 2x - \sin 10x \sin 2x$$

10. Prove the following non-famous identity.

$$\cos(8x) = \cos 4x \cos 4x - \sin 4x \sin 4x$$

11. Prove the following non-famous identity.

$$\cos 10x \cos 2x - \sin 10x \sin 2x = \cos 4x \cos 4x - \sin 4x \sin 4x$$

12. Without calculators determine if the following is true, then explain...

$$\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$$

13. Without calculators determine if the following is true, then explain...

$$\frac{\sqrt{6}+\sqrt{2}}{4} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

14. Prove and OWN everyone of these famous identities.

Double-Angle Identities

$$\cos(2a) = \cos^2 a - \sin^2 a \qquad \cos(2a) = 2\cos^2 a - 1$$
$$\cos(2a) = 1 - 2\sin^2 a \qquad \sin(2a) = 2\sin a \cos a$$
$$\tan(2a) = \frac{2\tan a}{1-\tan^2 a} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

15. Prove and OWN everyone of these famous identities.

Product-to-Sums Identities

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$
$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$
$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

16. Prove and OWN everyone of these famous identities.

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Sums-to-Products Identities

$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$
$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

17. (**)Prove the identity

$$\sin(2x) = 2\sin x\cos x$$