Famous IDs: Double-Angle Identities

Main Idea

We continue to expand the list of very famous trigonometric identities, and to practice our proving skills. Virtually all identities presented in the last section, this section and in the next few sections, ultimately, come from from the *mother of them all*.

Double-Angle Identities

$$\cos(2a) = \cos^2 a - \sin^2 a \qquad \cos(2a) = 2\cos^2 a - 1$$
$$\cos(2a) = 1 - 2\sin^2 a \qquad \sin(2a) = 2\sin a \cos a$$
$$\tan(2a) = \frac{2\tan a}{1-\tan^2 a} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

EXAMPLE 1 (by tweaking a known identity) Prove the following identity

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Solution: We will substitute, $a = x$ and $b = x$, into the $\cos(a + b)$ identity.

$\cos(a+b)$	$=$	$\cos a \cos b - \sin a \sin b$	Known Id, from sum-angle ids
$\cos(x+x)$	$=$	$\cos x \cos x - \sin x \sin x$	the Tweak, substitute $a = x$ and $b = x$
$\cos(2x)$	$=$	$\cos^2 x - \sin^2 x$	clean up.... done!

EXAMPLE 2 (by tweaking a known identity, the identity from example one.) Prove the following identity

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

Solution: We begin with a know identity identity.

$\cos(2x)$	$=$	$\cos^2 x - \sin^2 x$	known, proven above
$\cos(2x)$	$=$	$(1-\sin^2 x) - \sin^2 x$	substitute from the pythagoras ids $1 = \sin^2 x + \cos^2 x$
$\cos(2x)$	$=$	$1 - 2\sin^2 x$	algebra clean up
$2\sin^2 x$	$=$	$1 - \cos(2x)$	algebra clean up, trying to isolate $\sin^2 x$
$\sin^2 x$	$=$	$\dfrac{1-\cos(2x)}{2}$	algebra clean up,divide by 2, done!!

Famous IDs: Double-Angle Identities

1. Prove and OWN everyone of the famous identity:

$$\cos(2a) = \cos^2 a - \sin^2 a$$

> **Solution:** ONE way to prove $\cos(2a) = \cos^2 a - \sin^2 a$ is to start with MOTA,
>
> $$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
>
> Then the tweak, sub $x = a$ and $y = -a$

2. Prove and OWN everyone of the famous identity:

$$\cos(2a) = 2\cos^2 a - 1$$

> **Solution:** ONE way to prove $\cos(2a) = 2\cos^2 a - 1$, start with $\cos(2a) = \cos^2 a - \sin^2 a$ ... then the tweak, sub from the pythagoras family of identities... $\sin^2 a = 1 - \cos^2 a$

3. Prove and OWN everyone of the famous identity:

$$\cos(2a) = 1 - 2\sin^2 a$$

> **Solution:** ONE way to prove $\cos(2a) = 1 - 2\sin^2 a$, start with $\cos(2a) = \cos^2 a - \sin^2 a$ ... then the tweak, sub from the pythagoras family of identities... $\cos^2 a = 1 - \sin^2 a$

4. Prove and OWN everyone of the famous identity:

$$\sin(2a) = 2\sin a \cos a$$

> **Solution:** ONE way to prove $\sin(2a) = 2\sin a \cos a$, start with the sum-angle identity for $\sin(x + y)$ ... then the tweak, sub $x = a$ and $y = a$

5. Prove and OWN everyone of the famous identity:

$$\tan(2a) = \frac{2\tan a}{1 - \tan^2 a}$$

> **Solution:** ONE way to prove $\tan(2a) = \frac{2\tan a}{1-\tan^2 a}$, start with the sum-angle identity for $\tan(x + y)$ ... then the tweak, sub $x = a$ and $y = a$

6. Prove and OWN everyone of the famous identity:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

> **Solution:** ONE way to prove $\sin^2 a = \frac{1-\cos(2a)}{2}$, start with $\cos(2a) = 1 - 2\sin^2 a$ then the tweak, *solve* for $\sin^2 a$

7. Prove and OWN everyone of the famous identity:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

> **Solution:** ONE way to prove $\cos^2 a = \frac{1+\cos(2a)}{2}$, start with $\cos(2a) = 2\cos^2 a - 1$ then the tweak, *solve* for $\cos^2 a$

8. Prove the following non-famous identity.

$$\cos(4x) = \cos^2(2x) - \sin^2(2x)$$

> **Solution:** ONE way to prove $\cos(4x) = \cos^2(2x) - \sin^2(2x)$ is to start with $\cos(2a) = \cos^2 a - \sin^2 a$ Then the tweak, sub $a = 2x$

9. Prove the following non-famous identity.

$$\cos(10x) = \cos^2(5x) - \sin^2(5x)$$

> **Solution:** ONE way to prove this is to start with $\cos(2a) = \cos^2 a - \sin^2 a$ Then the tweak, sub $a = 5x$

10. Prove the following non-famous identity.

$$\cos\left(\frac{x}{3}\right) = \cos^2\left(\frac{x}{6}\right) - \sin^2\left(\frac{x}{6}\right)$$

> **Solution:** ONE way to prove this is to start with $\cos(2a) = \cos^2 a - \sin^2 a$ Then the tweak, sub $a = \frac{x}{6}$

11. Prove the following non-famous identity.

$$\sin(6x) = 2\sin 3x\cos 3x$$

> **Solution:** ONE way to prove this is to start with $\sin(2a) = 2\sin a\cos a$ Then the tweak, sub $a = 3x$

12. Prove the following non-famous identity.

$$\sin(10x) = 2\sin 5x\cos 5x$$

Solution: ONE way to prove this is to start with $\sin(2a) = 2\sin a \cos a$ Then the tweak, sub $a = 5x$

13. Prove the following non-famous identity.

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

Solution: ONE way to prove this is to start with $\cos(2a) = \cos^2 a - \sin^2 a$ Then the tweak, sub $a = \frac{x}{2}$

14. Prove the following non-famous identity.

$$\tan(10x) = \frac{2\tan 5x}{1 - \tan^2 5x}$$

15. Prove the following non-famous identity.

$$\cos(4x) = 1 - 2\sin^2(2x)$$

16. Prove the following non-famous identity.

$$\cos(4x) = 1 - 2\sin^2 x \cos^2 x$$

17. Without calculators determine if the following is true, then explain...

$$\cos 15^\circ = \sqrt{\frac{2+\sqrt{3}}{4}}$$

Solution: start with a known identity, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, then substitute $\theta = 15^\circ$