Famous IDs: Sum & Products Identities

Main Idea

We continue to expand the list of very famous trigonometric identities, and to practice our proving skills. Virtually all identities presented in the last section, this section and in the next few sections, ultimately, come from from the *mother of them all.*

Product-to-Sums Identities

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$
$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$
$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

Sums-to-Products Identities

$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$
$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

EXAMPLE 1 (by working on the right hand side) Prove the following identity

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

Solution: We will work on the right hand side using the identities we have already proven for $\cos(a-b)$ and $\cos(a+b)$

$\sin a \sin b$	$\stackrel{?}{=}$	$\frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$	want to know
	|	$\frac{1}{2}\left[\cos a\cos b + \sin a \sin b - (\cos a \cos b - \sin a \sin b)\right]$	proven identities
	|	$\frac{1}{2}\left[\cos a\cos b + \sin a \sin b - \cos a \cos b + \sin a \sin b)\right]$	distribute the negative
	|	$\frac{1}{2}\left[2\sin a \sin b\right]$	clean up
	$=$	$\sin a \sin b$	kachin! kachin!

EXAMPLE 2 (by tweaking a known identity, the identity from example one.) Prove the following identity

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Solution: We begin with a know identity identity, the one proven above. We then rearrange the equation to match the idenity we are trying to prove. The essential step is the substitution, the real tweak, substituting $a = \left(\frac{x+y}{2}\right)$ and $b = \left(\frac{x-y}{2}\right)$.

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \quad \text{(known, proven above)}$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \quad \text{(mult by 2, a little tweak)}$$

$$\cos(a+b) - \cos(a-b) = -2\sin a \sin b \quad \text{(move sides, another little tweak)}$$

$$\cos\left[\left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right)\right] - \cos\left[\left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right)\right] = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \quad \text{(the real tweak, sub.)}$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \quad \text{(clean up \& done!!)}$$

Famous IDs: Sum & Products Identities

1. Prove and OWN everyone of these famous identities.

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

Solution: Solution: We will work on the right hand side using the identities we have already proven for $\cos(a-b)$ and $\cos(a+b)$

$\sin a \sin b$	$\stackrel{?}{=}$	$\frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$	want to know
	|	$\frac{1}{2}\left[\cos a \cos b + \sin a \sin b - (\cos a \cos b - \sin a \sin b)\right]$	proven identities
	|	$\frac{1}{2}\left[\cos a \cos b + \sin a \sin b - \cos a \cos b + \sin a \sin b)\right]$	distribute the negative
	|	$\frac{1}{2}\left[2 \sin a \sin b\right]$	clean up
	$=$	$\sin a \sin b$	kachin! kachin!

2. Prove and OWN everyone of these famous identities.

$$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

Solution: Solution: We begin with a know identity identity, the one proven above. We then rearrange the equation to match the idenity we are trying to prove. The essential step is the substitution, the real tweak, substituting $a = \left(\frac{x+y}{2}\right)$
and $b = \left(\frac{x-y}{2}\right)$.

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \quad \text{(known, proven above)}$$
$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \quad \text{(mult by 2, a little tweak)}$$
$$\cos(a+b) - \cos(a-b) = -2\sin a \sin b \quad \text{(move sides, another little tweak)}$$
$$\cos\left[\left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right)\right] - \cos\left[\left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right)\right] = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
(the real tweak, sub.)
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \quad \text{(clean up \& done!!)}$$

3. Prove and OWN everyone of these famous identities.

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

Solution: Solution: We will work on the right hand side using the identities we have already proven for $\cos(a-b)$ and $\cos(a+b)$

$\cos a \cos b$	$\stackrel{?}{=}$	$\frac{1}{2}[\cos(a-b)+\cos(a+b)]$	want to know
	$\vert$	$\frac{1}{2}[\cos a\cos b+\sin a\sin b+\cos a\cos b-\sin a\sin b]$	proven identities
	$\vert$	$\frac{1}{2}[\cos a\cos b+\sin a\sin b-\cos a\cos b-\sin a\sin b)]$	distribute the negative
	$\vert$	$\frac{1}{2}[2\cos a\cos b]$	clean up
	$=$	$\cos a\cos b$	kachin! kachin!

4. Prove and OWN everyone of these famous identities.

$$\cos a+\cos b=2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Solution: Solution: We begin with a know identity identity, the one proven above. We then rearrange the equation to match the idenity we are trying to prove. The essential step is the substitution, the real tweak, substituting $a=\left(\frac{x+y}{2}\right)$ and $b=\left(\frac{x-y}{2}\right)$.

$$\frac{1}{2}[\cos(a-b)+\cos(a+b)]=\cos a\cos b \qquad \text{(known, proven above)}$$
$$\cos(a-b)+\cos(a+b)=2\cos a\cos b \qquad \text{(mult by 2, a little tweak)}$$
$$\cos(a+b)+\cos(a-b)=2\cos a\cos b \qquad \text{(prepare for substitution..)}$$
$$\cos\left[\left(\frac{x+y}{2}\right)+\left(\frac{x-y}{2}\right)\right]+\cos\left[\left(\frac{x+y}{2}\right)-\left(\frac{x-y}{2}\right)\right]=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \text{(the real tweak, sub.)}$$
$$\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \text{(clean up \& done!!)}$$

5. Prove and OWN everyone of these famous identities.

$$\sin a\cos b=\frac{1}{2}[\sin(a+b)+\sin(a-b)]$$

Solution: see solutions to ex. #1, 2, 3, &4

6. Prove and OWN everyone of these famous identities.

$$\sin a+\sin b=2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Solution: see solutions to ex. #1, 2, 3, &4

7. Prove and OWN everyone of these famous identities.

$$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

Solution: see solutions to ex. #1, 2, 3, &4

8. Prove the following non-famous identity.

$$2\cos(4x)\cos(3x) = \cos x + \cos(7x)$$

Solution:

	$\cos a\cos b = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$	(already proven)
the tweak, sub $a = 4x$ and $b = 3x$	$\cos 4x\cos 3x = \frac{1}{2}\left[\cos(4x-3x)+\cos(4x+3x)\right]$	(sub)
	$2\cos 4x\cos 3x = \cos(x)+\cos(7x)$	(algebra)

9. Prove the following non-famous identity.

$$\cos(4x) + \cos(3x) = 2\cos(3.5x)\cos(.5x)$$

Solution:

	$\cos a\cos b = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$	(already proven)
the tweak, sub $a = 3.5x$ and $b = .5x$	$\cos 3.5x\cos .5x = \frac{1}{2}\left[\cos(3.5x-.5x)+\cos(3.5x+.5x)\right]$	(sub)
	$2\cos 3.5x\cos .5x = \cos(3x)+\cos(4x)$	(algebra)

10. Prove the following non-famous identity.

$$\sin(4x) + \sin(6x) = 2\sin(5x)\cos(x)$$

Solution:

	$\sin a\cos b = \frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$	(already proven)
the tweak, sub $a = 5x$ and $b = x$	$\sin 5x\cos x = \frac{1}{2}\left[\sin(5x+x)+\sin(5x-x)\right]$	(sub)
	$2\sin 5x\cos x = \sin(6x)+\sin(4x)$	(algebra)

11. Prove the following non-famous identity.

$$2(2\cos^2(2x) - 1)\cos(3x) - \cos(7x) = \cos x$$

Solution: First note $\cos(2a) = 2\cos^2 a - 1$ is a proven and famous identity, sub $a = 2x$ to recognize the variation, $2\cos^2(2x) - 1 = \cos(4x)$.... then.. work on left side...

$$2(2\cos^2(2x) - 1)\cos(3x) - \cos(7x) = 2\cos(4x)\cos(3x) - \cos(7x)$$

now from ex. #8 we see that $2\cos(4x)\cos(3x) = \cos x + \cos(7x)$ therefore...

$$\begin{aligned} 2(2\cos^2(2x) - 1)\cos(3x) - \cos(7x) &= 2\cos(4x)\cos(3x) - \cos(7x) && \text{(con't..)} \\ &= \cos x + \cos(7x) - \cos(7x) && \text{(sub)} \\ &= \cos x \end{aligned}$$

12. Without calculators determine if the following is true, then explain...

$$\frac{\sqrt{2} + \sqrt{3}}{2} = 2\cos(37.5^\circ)\cos(7.5^\circ)$$

Solution:

$$\begin{aligned} \cos a \cos b &= \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] && \text{(already proven)} \\ \cos 37.5^\circ \cos 7.5^\circ &= \frac{1}{2}\left[\cos(37.5^\circ - 7.5^\circ) + \cos(37.5^\circ + 7.5^\circ)\right] && \text{(sub)} \\ 2\cos 37.5^\circ \cos 7.5^\circ &= \cos(30^\circ) + \cos(45^\circ) && \text{(algebra)} \\ 2\cos 37.5^\circ \cos 7.5^\circ &= \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} && \text{(algebra)} \end{aligned}$$

the tweak, sub $a = 37.5^\circ$ and $b = 7.5^\circ$

13. (**)Prove the following non-famous identity.

$$4\cos^2(2x)\cos(3x) - 2\cos 3x - \cos(7x) = \cos x$$

Solution: work on left side... notice...

$$4\cos^2(2x)\cos(3x) - 2\cos 3x - \cos(7x) = 2\left[2\cos^2(2x) - 1\right]\cos(3x) - \cos(7x)$$

then continue as in ex. #11

