Euler's Identity

The Easy World of Exponents

According to the legend, John Napier, invented logs to rid science of all difficult computations. He was clearly thinking about the difference in difficulty of multiplying for example, $81 \cdot 27$ vs. multiplying $3^4 \cdot 3^3$, clearly the second one is easier, 3^7 since one simply adds the exponents when multiplying two numbers with the same base. Dividing, taking powers of, or roots is just as easy, for example $\sqrt{3^{10}} = 3^5$. We take a moment here to recall some of these famous exponent properties, for any real numbers, a, b, m, n with $a, b > 0$:

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

and

$$(ab)^m = a^m b^m$$

Compare the worlds: Exponents are Easier

	Harder	*Easier*
Multiply numbers	$32 \cdot 64$	$2^5 \cdot 2^6 = 2^{11}$
Divide numbers	$\frac{243}{27}$	$\frac{3^5}{3^3} = 3^2$
Powers of numbers	$(243)^6$	$\left(3^5\right)^6 = 3^{30}$
Roots of numbers	$(256)^{1/2}$	$(2^8)^{1/2} = 2^4$

The Guinus of Euler

Before Euler, integer and rational exponents were common and understood by most. Even irrational exponents were understood and well within what most considered reasonable and ordinary. But *no one human being* in the history of our civilization had ever the audacity, creativity, and guts to try a non-real exponent, such as 2^i or 2^{3+5i}? Euler did. Not only did he ask such a question, he answered it. Not only did he answer it, he answered it with the most elegant and amazing identity. Regarded by many as one of the top 3 identities of all times, all languages, all worlds, is Eulers Identity.

$$\boxed{e^{i\theta} = \cos\theta + i \sin\theta}$$

Eulers Identity makes a bridge between every complex number, from standard form $a + bi$, to a number of the form $re^{i\theta}$. The great advantage is that multiplying, dividing, taking powers, and finding roots all becomes childs play when numbers are written in Eulers form[0], $re^{i\theta}$.

Multiplying in the World of Euler

$$e^{i30^\circ} \cdot e^{i60^\circ} = e^{i90^\circ}$$

Dividing in the World of Euler

$$\frac{e^{i30^\circ}}{e^{i60^\circ}} = e^{-i30^\circ}$$

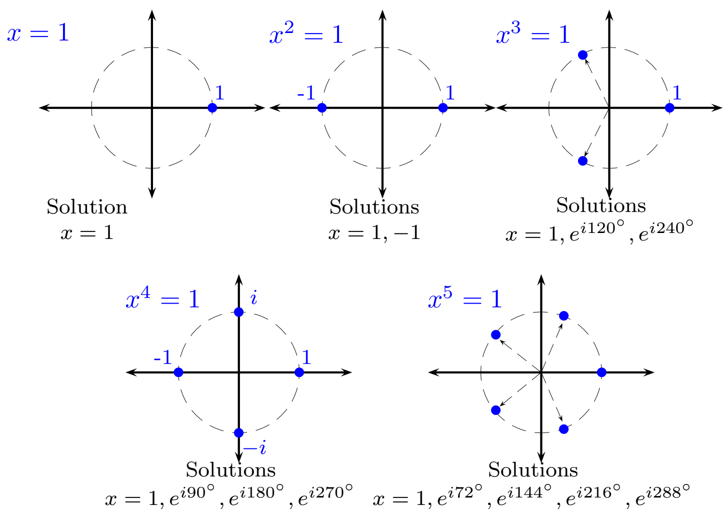
Powers in the World of Euler

$$\left(e^{i30^\circ}\right)^4 = e^{i120^\circ}$$

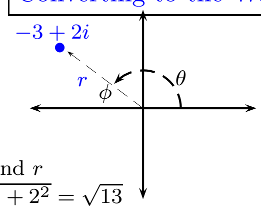
The Fundamental Theorem of Algebra (F.T.A.)

The Fundamental Theorem of Algebra says a polynomial of degree 1 has exactly one root, a polynomial of degree 2 has exaclty two roots if multiplicity is counted, a polynomial of degree 3 has exaclty three roots if multiplicity is counted, etc. etc... For example, $x = 1$ has one solution, $x^2 = 1$ has two, $x^3 = 1$ has three, etc.etc...

The Roots of Unity and F.T.A.



Converting to the World of Euler



to find r

$$r = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

to find θ[1]

$\theta \approx 146.7^\circ$

$\phi = \tan^{-1}(2/3)$

$\approx 33.7^\circ$

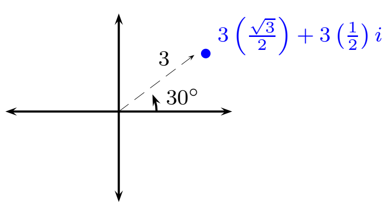
Euler's Identity

1. **Convert** Convert $3e^{i30°}$ to standard form, $a + bi$

Solution: According to Euler's amazing identity..$e^{i\theta} = \cos\theta + i\sin\theta$..thus..

$$\begin{aligned} 3e^{i30°} &= 3[\cos 30° + i\sin 30°] \\ &= 3\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] \\ &= \frac{3\sqrt{3}}{2} + i\frac{3}{2} \end{aligned}$$

Alternatively, you could convert using the picture... and convert in very much the same way we convert from polar to cartesian...



2. **Convert** Convert $3e^{-i30°}$ to standard form, $a + bi$

Solution: According to Euler's amazing identity..$e^{i\theta} = \cos\theta + i\sin\theta$..thus..

$$\begin{aligned} 3e^{-i30°} &= 3[\cos -30° + i\sin -30°] \\ &= 3\left[\frac{\sqrt{3}}{2} + i\frac{-1}{2}\right] \\ &= \frac{3\sqrt{3}}{2} + \frac{-3}{2}i \end{aligned}$$

Or use the picture... Or use polar conversion ideas.. $(r, \theta) = (3, -30°)$

3. **Convert** Convert $e^{i\frac{\pi}{2}}$ to standard form, $a + bi$

Solution: According to Euler's amazing identity..$e^{i\theta} = \cos\theta + i\sin\theta$..thus..

$$\begin{aligned} e^{i\frac{\pi}{2}} &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i(1) \\ &= i \end{aligned}$$

4. **Convert** Convert $e^{\frac{3\pi}{2}i}$ to standard form, $a + bi$
5. **Convert** Convert $e^{\frac{7\pi}{6}i}$ to standard form, $a + bi$
6. **Convert** Convert $3e^{2i}$ to standard form, $a + bi$
7. **Convert** Convert $5e^{3.14i}$ to standard form, $a + bi$
8. **Convert** Convert $e^{6.28i}$ to standard form, $a + bi$
9. **Convert** Convert $2 + i$ to Euler form, $re^{i\theta}$

Solution: $2 + i = \sqrt{5}e^{i26.57^\circ}$ many other correct possibilities for theta..

10. **Convert** Convert $2 - 2i$ to Euler form, $re^{i\theta}$

Solution: $2 - 2i = (2\sqrt{2})e^{-i45^\circ}$ OR $(2\sqrt{2})e^{i315^\circ}$ or $(2\sqrt{2})e^{i765^\circ}$ or infinite many more possibilities for θ

11. **Convert** Convert $-2 - 2i$ to Euler form, $re^{i\theta}$

Solution: $-2 - 2i = (2\sqrt{2})e^{i225^\circ}$ many other correct possibilities for theta..

12. **Convert** Convert $-3 + 2i$ to Euler form, $re^{i\theta}$
13. **Convert** Convert i to Euler form, $re^{i\theta}$

Solution: $i = e^{i90^\circ}$ many other correct possibilities for theta.. note $r = 1$

14. **Convert** Convert $3i$ to Euler form, $re^{i\theta}$

Solution: $i = 3e^{i90^\circ}$ many other correct possibilities for theta.. note $r = 1$

15. **Convert** Convert $.5 + 3i$ to Euler form, $re^{i\theta}$
16. **Convert** Convert -1 to Euler form, $re^{i\theta}$

Solution: $i = e^{i180^\circ}$ many other correct possibilities for theta.. note $r = 1$

17. **Convert** Convert $-i$ to Euler form, $re^{i\theta}$

Solution: $i = e^{i270^\circ}$ many other correct possibilities for theta.. note $r = 1$

18. **Euler's World** Multiply or divide as indicated

(a) Multiply the old fashion way...

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

(b) Multiply the Euler way, and simplify answer.. (note these are the same numbers..)

$$e^{30^\circ i} \cdot e^{60^\circ i}$$

Solution: $e^{i90^\circ} = i$

19. **Euler's World** Multiply or divide as indicated

(a) Multiply the old fashion way...

$$\left(\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right) \cdot \left(\frac{2}{2} + \frac{2\sqrt{3}}{2}i\right)$$

(b) Multiply the Euler way, and simplify answer.. (note these are the same numbers..)

$$3e^{30^\circ i} \cdot 2e^{60^\circ i}$$

Solution: $6e^{i90^\circ} = 6i$

20. **Euler's World** Multiply or divide as indicated

(a) Multiply the old fashion way...

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

(b) Multiply the Euler way, and simplify answer.. (note these are the same numbers..)

$$e^{30^\circ i} \cdot e^{30^\circ i}$$

Solution: e^{i60°

21. **Euler's World** Multiply or divide as indicated

(a) Divide the old fashion way...

$$\frac{\frac{\sqrt{3}}{2} + \frac{1}{2}i}{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

(b) Divide the Euler way, and simplify answer.. (note these are the same numbers..)

$$\frac{e^{30^\circ i}}{e^{60^\circ i}}$$

Solution: e^{-i30°

22. **Euler's World** Multiply or divide as indicated

(a) Expand the binomial, the old fashion way...

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3$$

(b) Calculate, the Euler way, and simplify answer.. (note these are the same numbers..)

$$\left(e^{30^\circ i}\right)^3$$

Solution: $e^{i90^\circ} = i$ compare this problem with #57 from two sections ago.

23. **Euler's World** Multiply or divide as indicated

(a) Expand the binomial, the old fashion way (or not..) ...

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^7$$

(b) Calculate, the Euler way, and simplify answer.. (note these are the same numbers..)

$$\left(e^{30^\circ i}\right)^7$$

Solution: e^{i210°

24. **Euler's World** Convert the numbers to Euler form, then compute.

(a)

$$(3 + 4i)(-2 - 5i)$$

(b)

$$\frac{3 + 4i}{-2 - 5i}$$

25. **Euler's World** to find Roots

(a) Try to find a square root of i, i.e. a number x such that $x^2 = i$

$$\sqrt{i}$$

(b) Find a root, the Euler way, and simplify answer.. (note these are the same numbers..)

$$\left(e^{90^\circ i}\right)^{\frac{1}{2}}$$

Solution: e^{i45°

26. **Euler's World** to find Roots

(a) Try to find a square root: (after a good honest attempt, go to part b.)

$$\sqrt{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

(b) Find a root, the Euler way, and simplify answer.. (note these are the same numbers..)

$$\left(e^{60^\circ i}\right)^{\frac{1}{2}}$$

Solution: e^{i30°

27. **Euler's World** to find Roots

(a) Try to find a fourth root: (after a good honest attempt, go to part b.)

$$\sqrt[4]{\frac{16}{2} + \frac{16\sqrt{3}}{2}i}$$

(b) Find a fourth root, the Euler way, and simplify answer.. (note these are the same numbers..)

$$\left(16e^{60^\circ i}\right)^{\frac{1}{4}}$$

Solution: $2e^{i15^\circ}$

28. **FTA and Unity Roots** The two solutions to $x^2 = 1$ are $x = 1, -1$. Show how one can derive these solutions.

29. **FTA and Unity Roots** The three solutions to $x^3 = 1$ are $x = 1$, $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{-1}{2} - \frac{\sqrt{3}}{2}i$. Show how one can derive these solutions.

30. **FTA and Unity Roots** The four solutions to $x^4 = 1$ are $x = 1, i, -1, -i$. Show how one can derive these solutions.

31. **FTA and Unity Roots** To find the five solutions to $x^5 = 1$, start with writing $1 = e^{k360^\circ i}$, then start finding 5th roots, every value of k should produce one...

Solution: for integers k, e^{ik72°

32. **Euler's World** Another Proof of MOTA

(a) Multiply the old fashion way

$$(\cos x + i \sin x)(\cos y + i \sin y)$$

(b) multiply, the Euler way, and simplify answer.. (note these are the same numbers..)

$$e^{ix} \cdot e^{iy}$$

(c) Compare the real part of the answer for a, with the real part of the answer for part b.