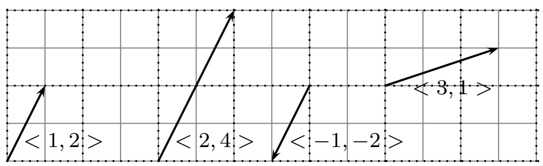
Vectors Introduction

What are vectors

Each vector is defined by two pieces of information: Direction and Magnitude. Often vectors are described by a picture representation or by ordered pairs which describe the direction and magnitude of the vector. To distinguish from the ordered pairs describing a point, vectors are written using pointy brackets rather than parenthesis. Variables representing vectors are often written in bold or with a hat or arrow over them. Here are a few examples.

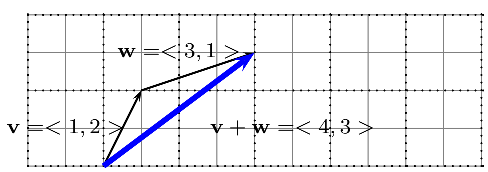
Examples of Vectors



Adding Vectors

To add two vectors, we simply add the corresponding components together. Using the diagrams to represent vectors we add by placing one vector at the tip of another, tip-to-tail. The vector uniting the starting point to the end point is commonly referred to as the resultant vector or the sum of the vectors. In some sense, the picture of addition of two vectors shows why vectors were invented to begin with. Their roots are in physics, perhaps the most famous example of a vector is a vector representing some Force. because Force is composed to two pieces of information, direction and magnitude, vectors are tailor made to represent such concepts. The direction of the force is represented by the direction of the vector while the magnitude of the force is represented by the magnitude of the vector. With this in mind, adding vectors works just as you may expect if you were adding two forces, tip-to-tail on the picture, component-wise algebraically. For Example:

$$< 1, 2 > + < 3, 1 > = < 4, 3 >$$



Normalizing of Vectors

To normalize a vector $\mathbf{v}$ refers to producing a new vector $\mathbf{n}_v$ such that this new vector has the same direction as $\mathbf{v}$ BUT magnitude exactly equal to one. Any non-zero vector can be normalized by simply multiplying by a scalar. Such scalar is $1/magnitude_v$. In other words,

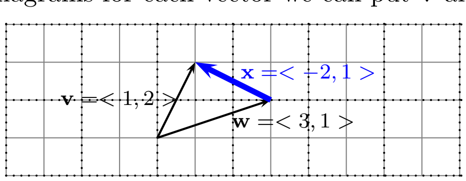
$$\mathbf{n}_v = \frac{1}{||\mathbf{v}||} \cdot \mathbf{v}$$

Subtracting Vectors

The key idea to subtract vectors is to turn the subtraction question into an addition question. For example suppose we want to find $\mathbf{v} - \mathbf{w}$. We can call such vector $\mathbf{x} = \mathbf{v} - \mathbf{w}$. Then we add $\mathbf{w}$ to both sides to get

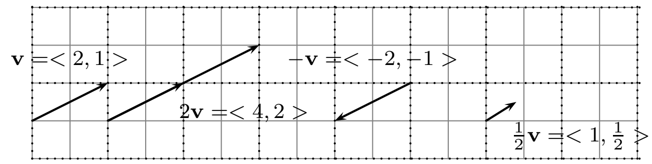
$$\mathbf{x} + \mathbf{w} = \mathbf{v}$$

Using the diagrams for each vector we can put $\mathbf{v}$ and $\mathbf{w}$, tail-to tail and solve for x as follows.



Scalar Multiples of Vectors

Generally, we define the scalar of a vector to be a vector with the same direction (or opposite direction) and possible scaled to a different size. If c is a real number and $\mathbf{v} =< a, b >$ a vector, then $c\mathbf{v} = c < a, b >=< ca, cb >$ is referred to as $\mathbf{v}$ times the scalar c. Note, if the scalar is negative, it reverses the direction of the vector. Here are some examples of scalars time a vector $\mathbf{v}$

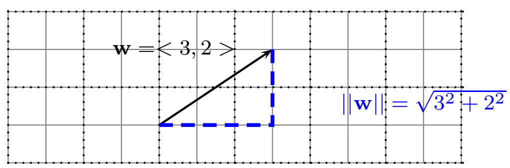


Magnitude of Vectors

Size, magnitude, and norm of a vector are often used interchangeably. In symbols, we use the 'double bars' to denote the magnitude of a vector.

$$||\mathbf{v}|| = \text{norm of } \mathbf{v}$$

To calculate the norm of a vector it is often helpful to draw a picture and pythagoras (i just made it a verb!) away[0]:



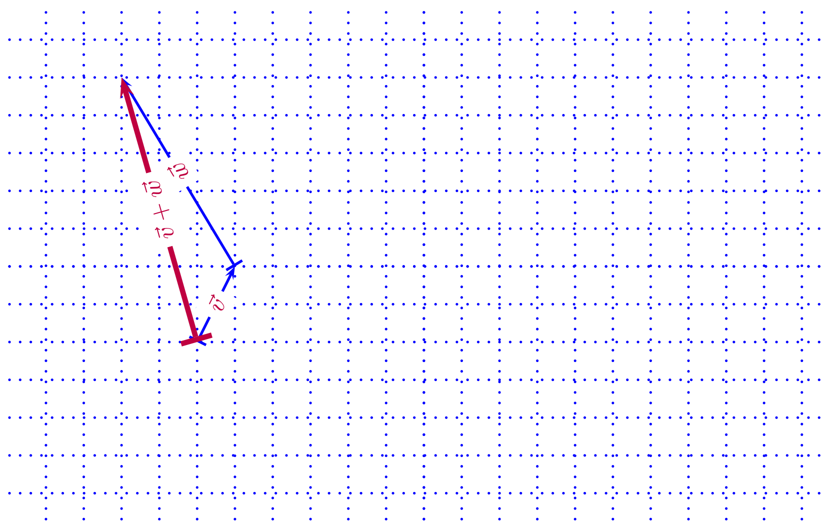
1. **Vector Arithmetic**

   (a) **Vector ADDITION** Suppose $\vec{v} = <1, 2>$ and $\vec{w} = <-3, 5>$, compute

$$\vec{v} + \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} + \vec{w} &= <1, 2> + <-3, 5> && \text{(given)} \\ &= <1 + -3, \quad 2 + 5> && \text{(definition of addition on vectors)} \\ &= <-2, 7> && \text{(by inspection)} \end{aligned}$$

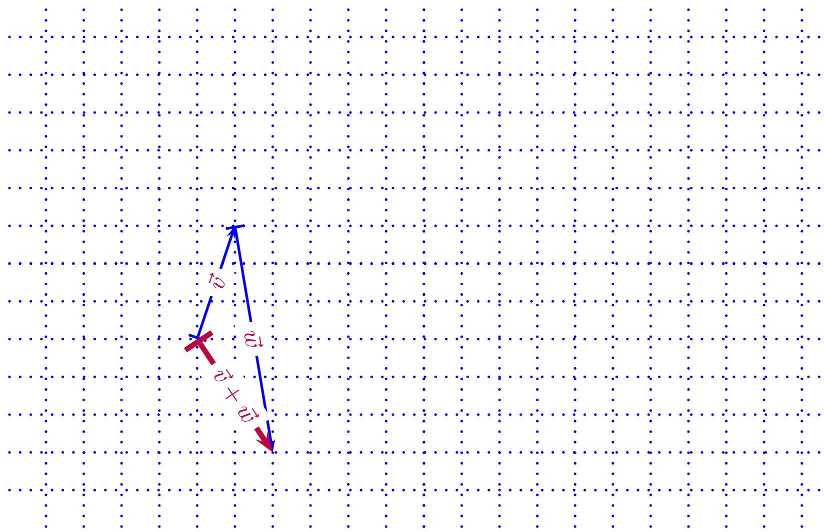


   (b) **Vector ADDITION** Suppose $\vec{v} = <1, 3>$ and $\vec{w} = <1, -6>$, compute

$$\vec{v} + \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} + \vec{w} &= <1, 3> + <1, -6> && \text{(given)} \\ &= <1 + 1, \quad 3 + -6> && \text{(definition of addition on vectors)} \\ &= <2, -3> && \text{(by inspection)} \end{aligned}$$

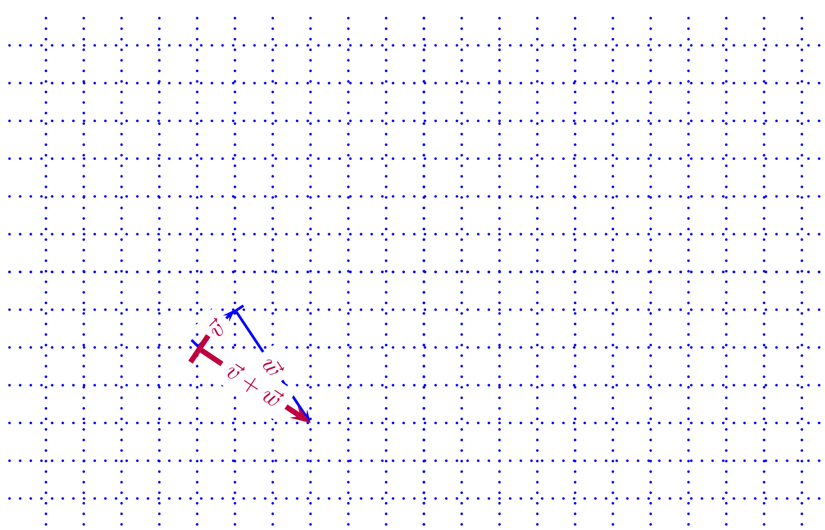


(c) **Vector ADDITION** Suppose $\vec{v} = <1,1>$ and $\vec{w} = <2,-3>$, compute

$$\vec{v}+\vec{w}$$

Solution:

$$\begin{aligned} \vec{v}+\vec{w} &= <1,1> + <2,-3> && \text{(given)} \\ &= <1+2, \quad 1+-3> && \text{(definition of addition on vectors)} \\ &= <3, -2> && \text{(by inspection)} \end{aligned}$$

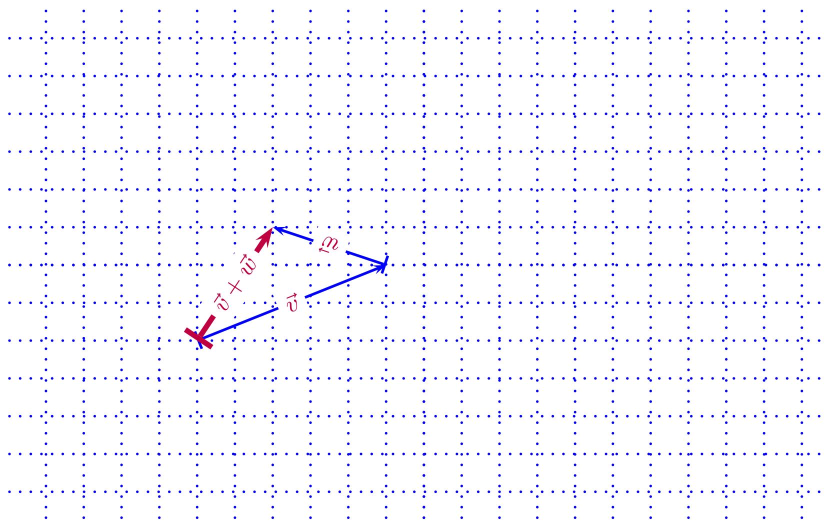


(d) **Vector ADDITION** Suppose $\vec{v} = <5,2>$ and $\vec{w} = <-3,1>$, compute

$$\vec{v}+\vec{w}$$

Solution:

$$\begin{aligned}\vec{v}+\vec{w} &= <5,2> + <-3,1> && \text{(given)}\\ &= <5+-3, \quad 2+1> && \text{(definition of addition on vectors)}\\ &= <2,3> && \text{(by inspection)}\end{aligned}$$

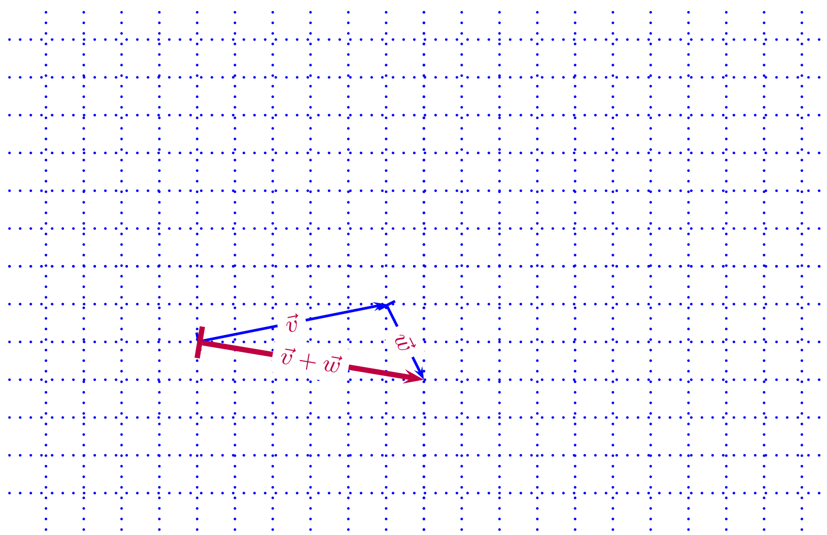


(e) **Vector ADDITION** Suppose $\vec{v} = <5,1>$ and $\vec{w} = <1,-2>$, compute

$$\vec{v}+\vec{w}$$

Solution:

$$\begin{aligned}\vec{v}+\vec{w} &= <5,1> + <1,-2> && \text{(given)}\\ &= <5+1, \quad 1+-2> && \text{(definition of addition on vectors)}\\ &= <6, -1> && \text{(by inspection)}\end{aligned}$$

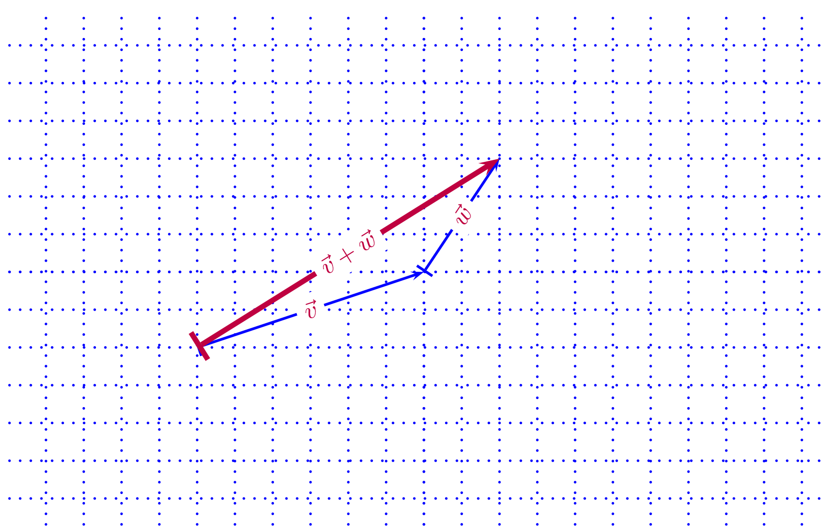


(f) **Vector ADDITION** Suppose $\vec{v} = <6, 2>$ and $\vec{w} = <2, 3>$, compute

$$\vec{v} + \vec{w}$$

Solution:

$$\begin{aligned} \vec{v}+\vec{w} &= <6,2> + <2,3> && \text{(given)} \\ &= <6+2, \quad 2+3> && \text{(definition of addition on vectors)} \\ &= <8,5> && \text{(by inspection)} \end{aligned}$$

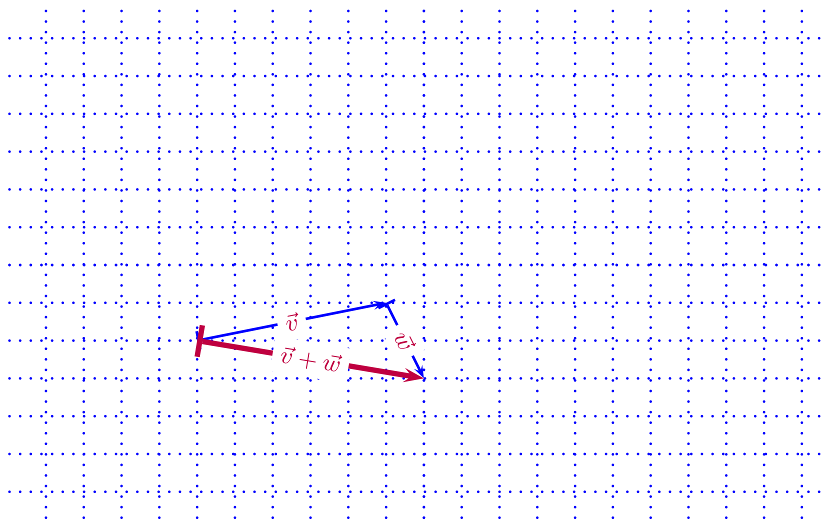


(g) **Vector ADDITION** Suppose $\vec{v} = <5, 1>$ and $\vec{w} = <1, -2>$, compute

$$\vec{v} + \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} + \vec{w} &= <5, 1> + <1, -2> && \text{(given)} \\ &= <5+1, \quad 1+-2> && \text{(definition of addition on vectors)} \\ &= <6, -1> && \text{(by inspection)} \end{aligned}$$

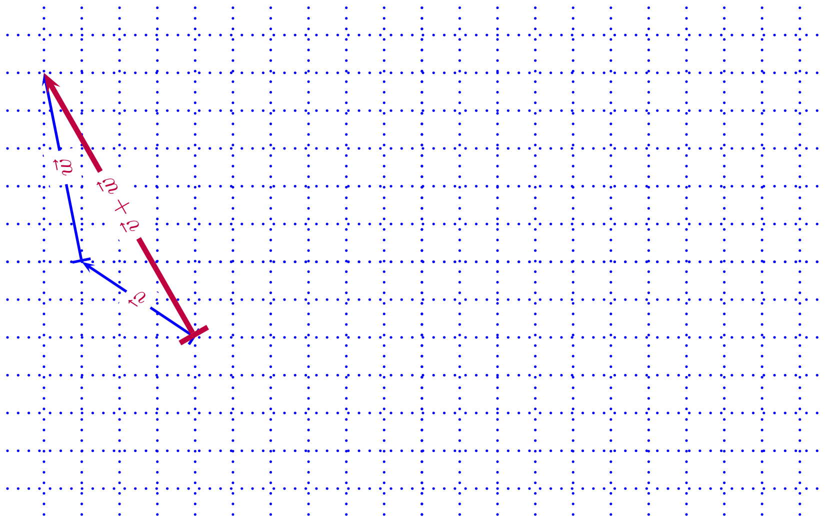


(h) **Vector ADDITION** Suppose $\vec{v} = <-3, 2>$ and $\vec{w} = <-1, 5>$, compute

$$\vec{v} + \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} + \vec{w} &= <-3, 2> + <-1, 5> && \text{(given)} \\ &= <-3+-1, \quad 2+5> && \text{(definition of addition on vectors)} \\ &= <-4, 7> && \text{(by inspection)} \end{aligned}$$



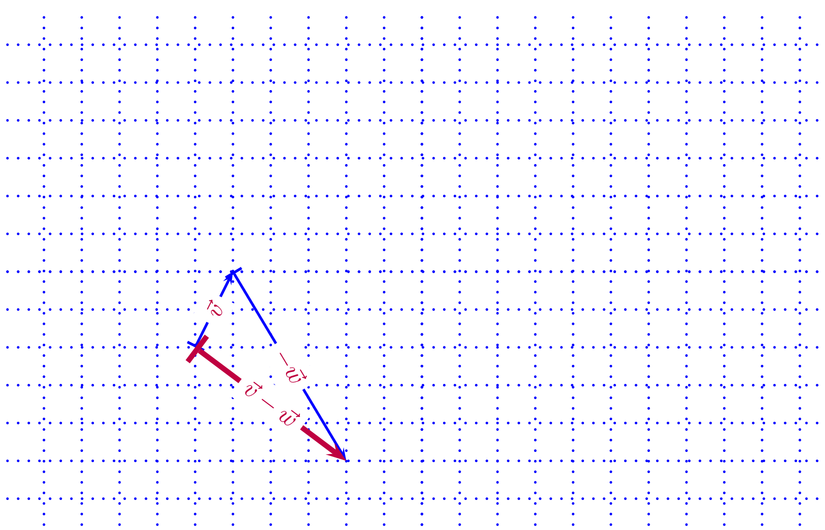
2. **Vector Arithmetic**

(a) **Vector SUBTRACTION** Suppose $\vec{v} = <1, 2>$ and $\vec{w} = < -3, 5>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} - \vec{w} &= <1, 2> - < -3, 5> && \text{(given)} \\ &= <1 - -3, \quad 2 - 5> && \text{(definition of subtraction on vectors)} \\ &= <4, -3> && \text{(by inspection)} \end{aligned}$$

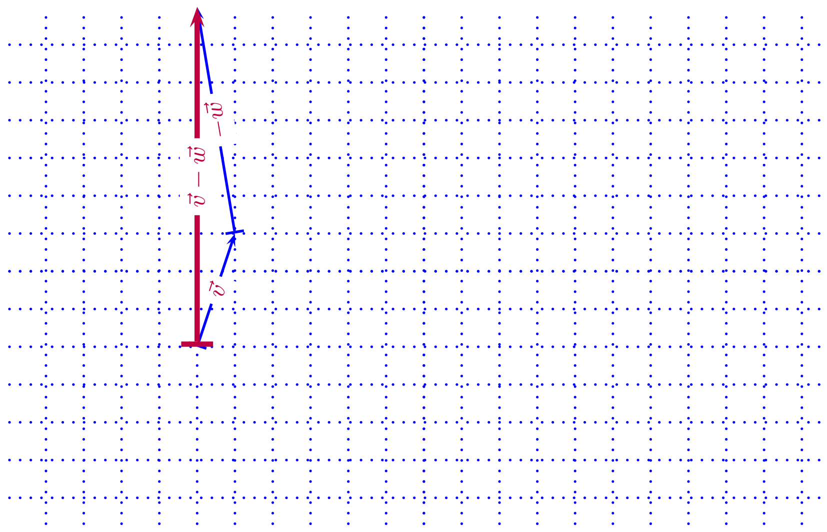


(b) **Vector SUBTRACTION** Suppose $\vec{v} = <1, 3>$ and $\vec{w} = <1, -6>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\vec{v} - \vec{w} = <1, 3> - <1, -6> \quad \text{(given)}$$
$$= <1 - 1, \quad 3 - -6> \quad \text{(definition of subtraction on vectors)}$$
$$= <0, 9> \quad \text{(by inspection)}$$

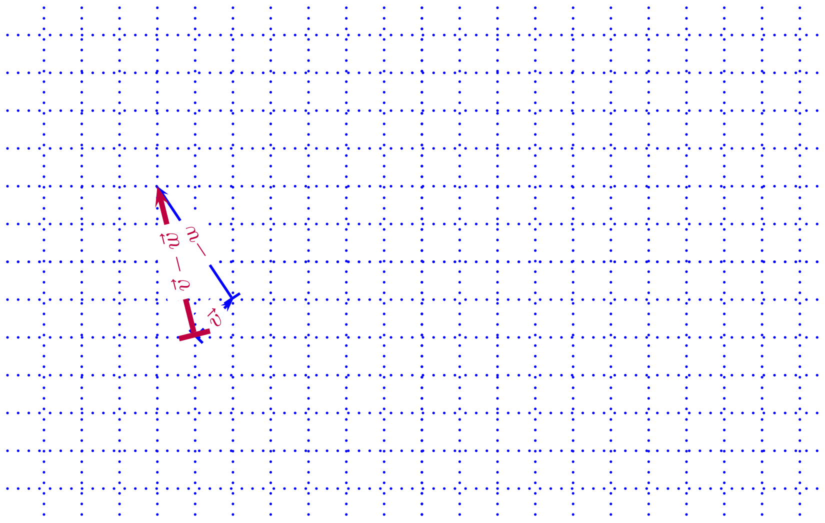


(c) **Vector SUBTRACTION** Suppose $\vec{v} = <1, 1>$ and $\vec{w} = <2, -3>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\vec{v} - \vec{w} = <1, 1> - <2, -3> \quad \text{(given)}$$
$$= <1 - 2, \quad 1 - -3> \quad \text{(definition of subtraction on vectors)}$$
$$= < -1, 4> \quad \text{(by inspection)}$$

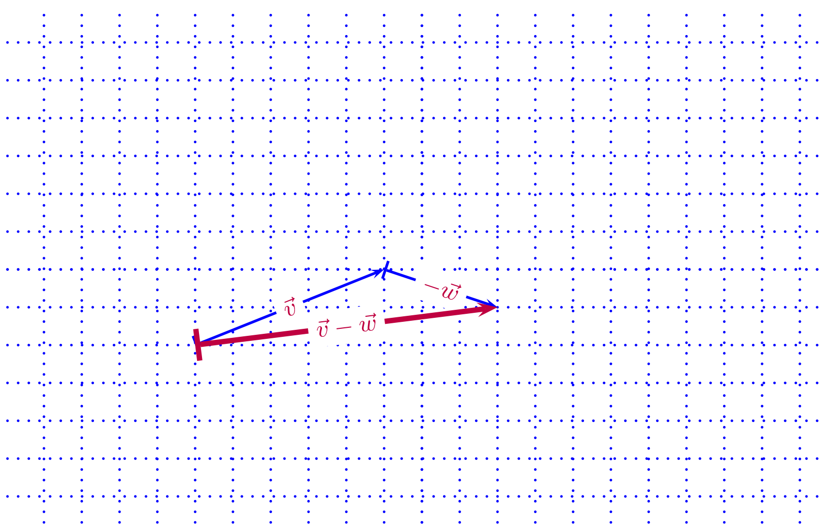


(d) **Vector SUBTRACTION** Suppose $\vec{v} = <5, 2>$ and $\vec{w} = < -3, 1>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} - \vec{w} &= <5, 2> - < -3, 1> && \text{(given)} \\ &= <5 - -3, \quad 2 - 1> && \text{(definition of subtraction on vectors)} \\ &= <8, 1> && \text{(by inspection)} \end{aligned}$$

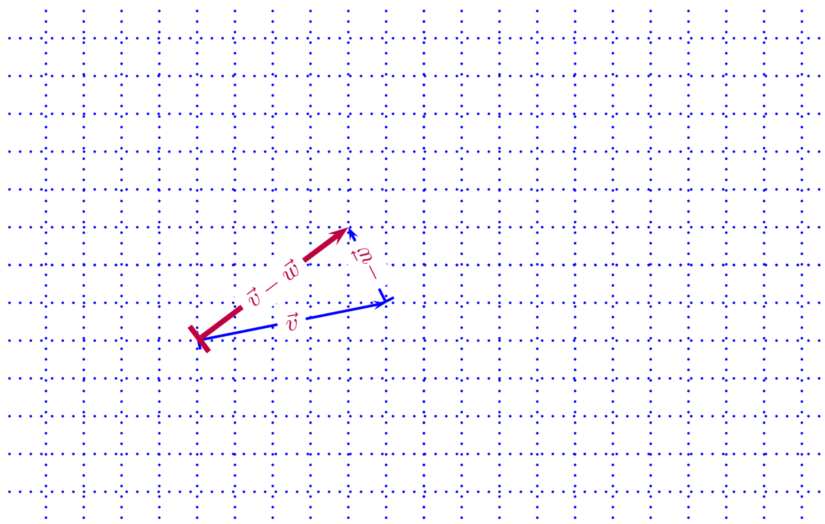


(e) **Vector SUBTRACTION** Suppose $\vec{v} = <5, 1>$ and $\vec{w} = <1, -2>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\begin{aligned}\vec{v} - \vec{w} &= <5, 1> - <1, -2> && \text{(given)}\\ &= <5 - 1, \quad 1 - -2> && \text{(definition of subtraction on vectors)}\\ &= <4, 3> && \text{(by inspection)}\end{aligned}$$

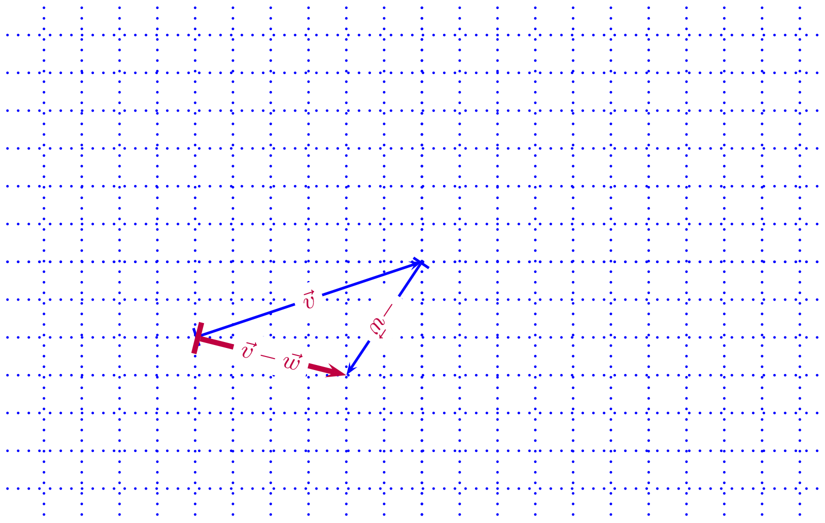


(f) **Vector SUBTRACTION** Suppose $\vec{v} = <6, 2>$ and $\vec{w} = <2, 3>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\begin{aligned}\vec{v} - \vec{w} &= <6, 2> - <2, 3> && \text{(given)}\\ &= <6 - 2, \quad 2 - 3> && \text{(definition of subtraction on vectors)}\\ &= <4, -1> && \text{(by inspection)}\end{aligned}$$

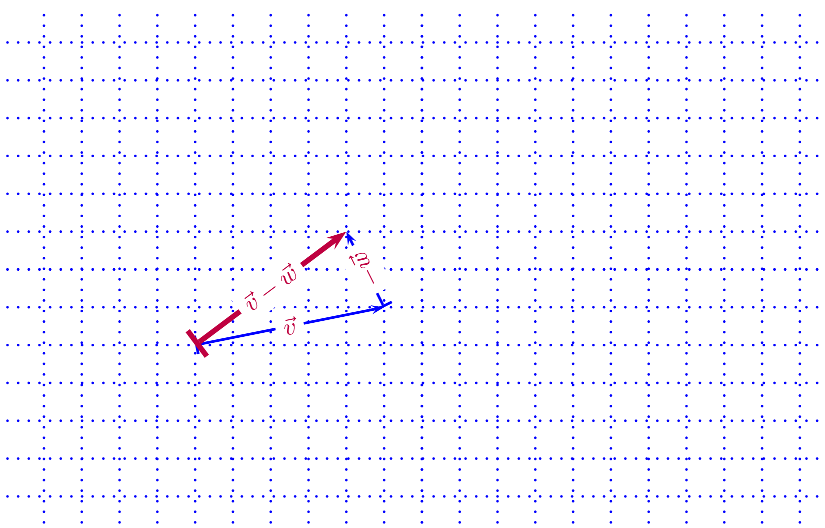


(g) **Vector SUBTRACTION** Suppose $\vec{v} = <5, 1>$ and $\vec{w} = <1, -2>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} - \vec{w} &= <5, 1> - <1, -2> && \text{(given)} \\ &= <5 - 1, \quad 1 - -2> && \text{(definition of subtraction on vectors)} \\ &= <4, 3> && \text{(by inspection)} \end{aligned}$$

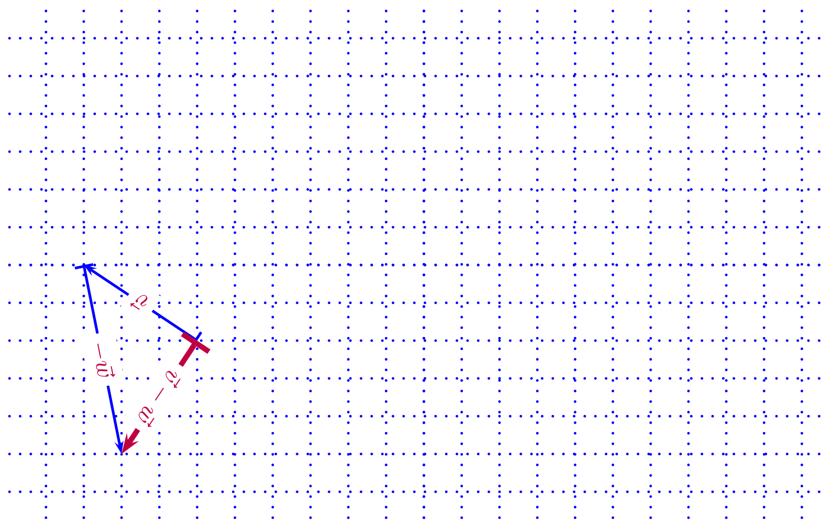


(h) **Vector SUBTRACTION** Suppose $\vec{v} = < -3, 2>$ and $\vec{w} = < -1, 5>$, compute

$$\vec{v} - \vec{w}$$

Solution:

$$\begin{aligned} \vec{v} - \vec{w} &= <-3, 2> - <-1, 5> && \text{(given)} \\ &= <-3 - -1, \quad 2 - 5> && \text{(definition of subtraction on vectors)} \\ &= <-2, -3> && \text{(by inspection)} \end{aligned}$$



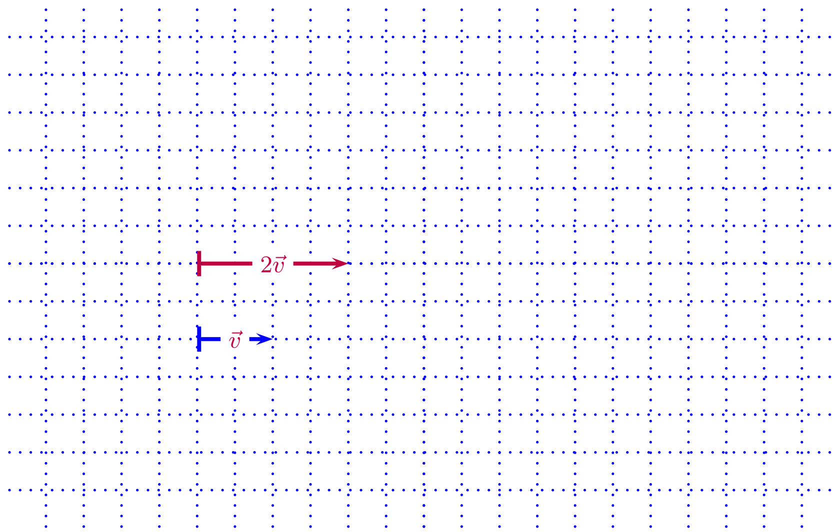
3. **Vector Arithmetic**

   (a) **Vector SCALARS** Suppose $\vec{v} = <2, 0>$compute

$$2\vec{v}$$

Solution:

$$\begin{aligned} 2\vec{v} &= 2<2, 0> && \text{(given)} \\ &= <2(2), 2(0)> && \text{(def of scalar multiplication)} \\ &= <4, 0> && \text{(by inspection)} \end{aligned}$$

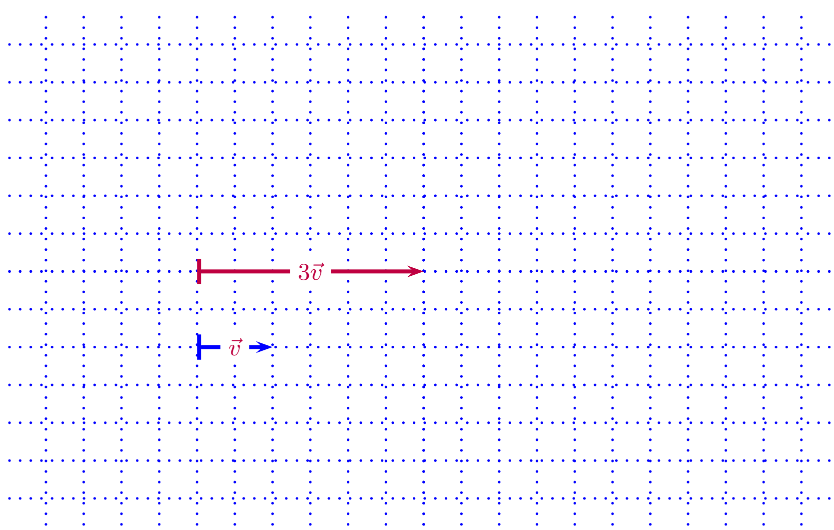


(b) **Vector SCALARS** Suppose $\vec{v} = <2, 0>$compute

$$3\vec{v}$$

Solution:

$$\begin{aligned} 3\vec{v} &= 3<2, 0> && \text{(given)} \\ &= <3(2), 3(0)> && \text{(def of scalar multiplication)} \\ &= <6, 0> && \text{(by inspection)} \end{aligned}$$

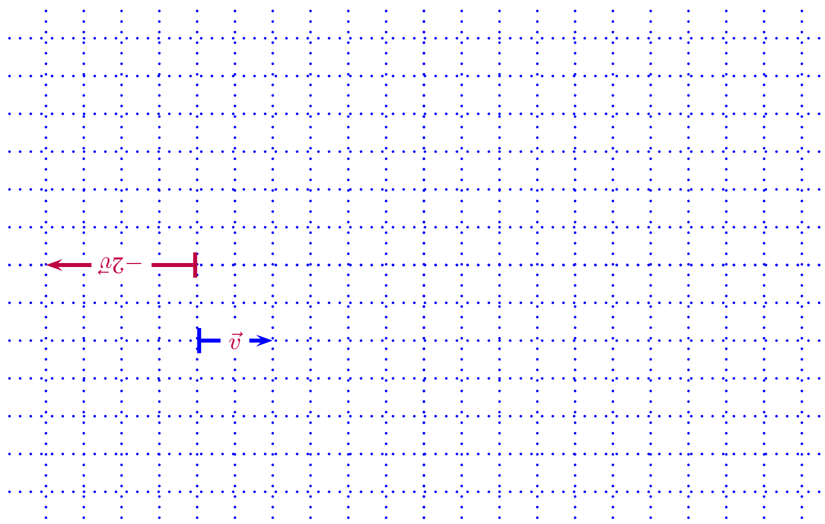


(c) **Vector SCALARS** Suppose $\vec{v} = <2, 0>$compute

$$-2\vec{v}$$

Solution:

$$
\begin{aligned}
-2\vec{v} &= -2<2,0> && \text{(given)}\\
&= <-2(2), -2(0)> && \text{(def of scalar multiplication)}\\
&= <-4,0> && \text{(by inspection)}
\end{aligned}
$$

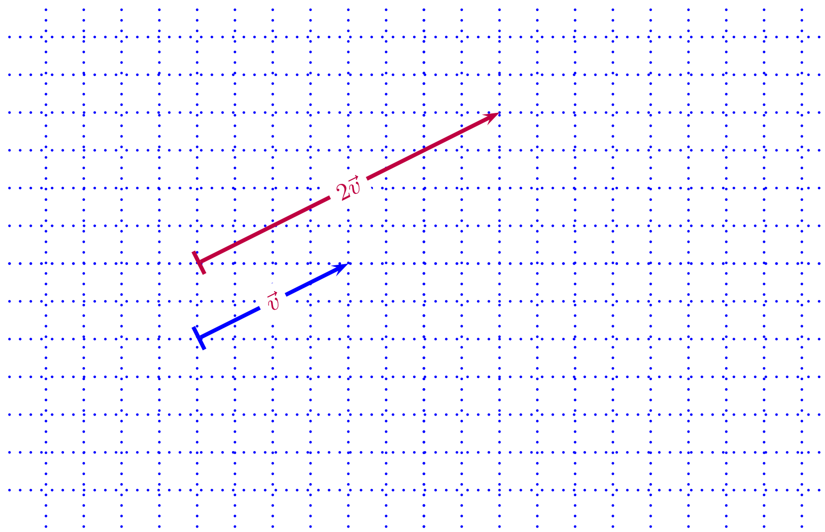


(d) **Vector SCALARS** Suppose $\vec{v} = <4,2>$compute

$$2\vec{v}$$

Solution:

$$
\begin{aligned}
2\vec{v} &= 2<4,2> && \text{(given)}\\
&= <2(4), 2(2)> && \text{(def of scalar multiplication)}\\
&= <8,4> && \text{(by inspection)}
\end{aligned}
$$

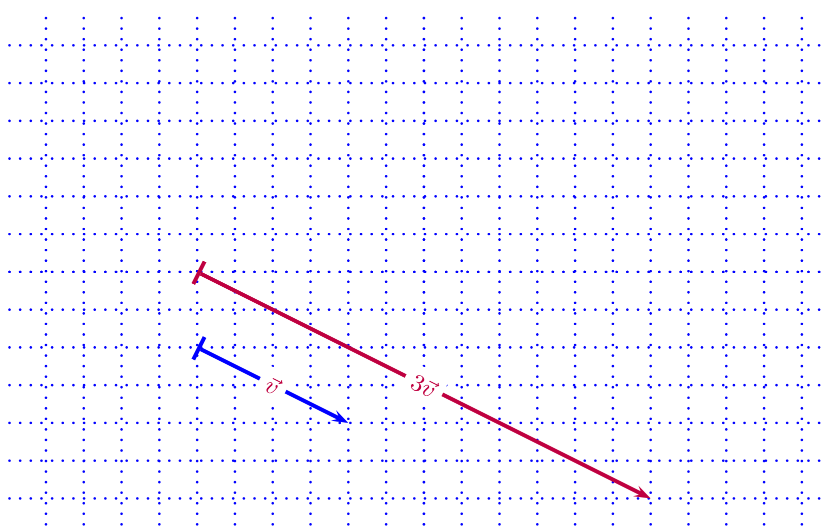


(e) **Vector SCALARS** Suppose $\vec{v} = <4, -2>$compute

$$3\vec{v}$$

Solution:

$$\begin{aligned} 3\vec{v} &= 3<4, -2> && \text{(given)} \\ &= <3(4), 3(-2)> && \text{(def of scalar multiplication)} \\ &= <12, -6> && \text{(by inspection)} \end{aligned}$$

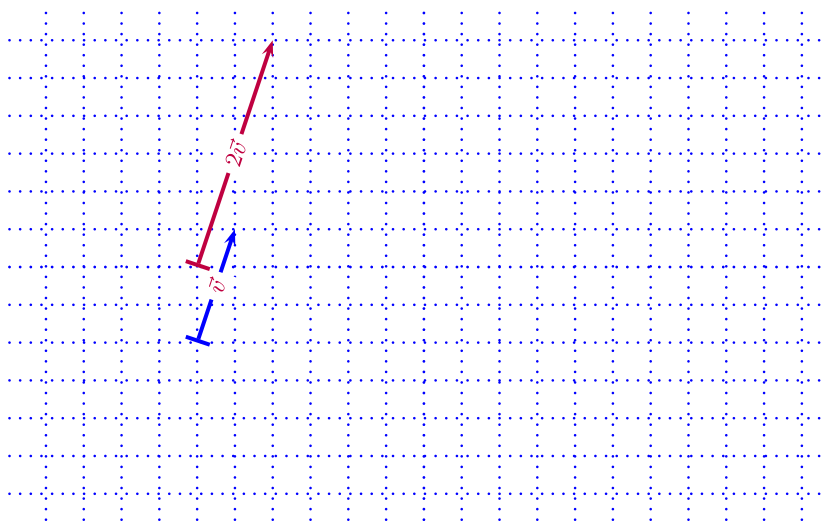


(f) **Vector SCALARS** Suppose $\vec{v} = <1, 3>$compute

$$2\vec{v}$$

Solution:

$$\begin{aligned} 2\vec{v} &= 2<1,3> && \text{(given)} \\ &= <2(1),2(3)> && \text{(def of scalar multiplication)} \\ &= <2,6> && \text{(by inspection)} \end{aligned}$$

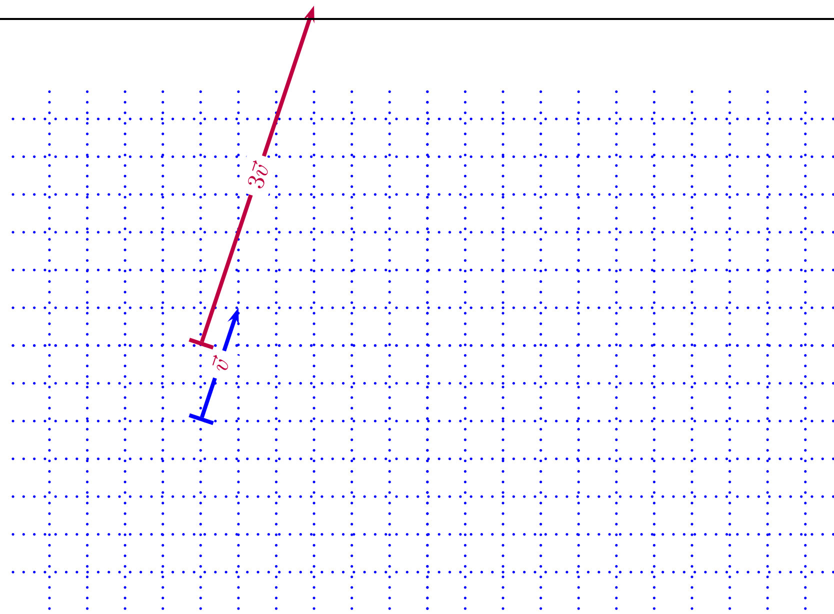


(g) **Vector SCALARS** Suppose $\vec{v} = <1,3>$compute

$$3\vec{v}$$

Solution:

$$\begin{aligned} 3\vec{v} &= 3<1,3> && \text{(given)} \\ &= <3(1),3(3)> && \text{(def of scalar multiplication)} \\ &= <3,9> && \text{(by inspection)} \end{aligned}$$

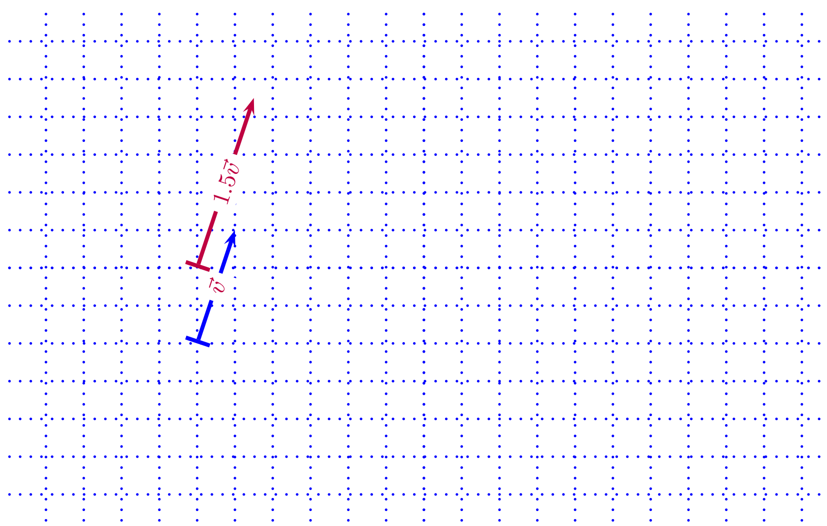


(h) **Vector SCALARS** Suppose $\vec{v} = <1, 3>$compute

$$1.5\vec{v}$$

Solution:

$$\begin{aligned} 1.5\vec{v} &= 1.5<1, 3> && \text{(given)} \\ &= <1.5(1), 1.5(3)> && \text{(def of scalar multiplication)} \\ &= <1.5, 4.5> && \text{(by inspection)} \end{aligned}$$

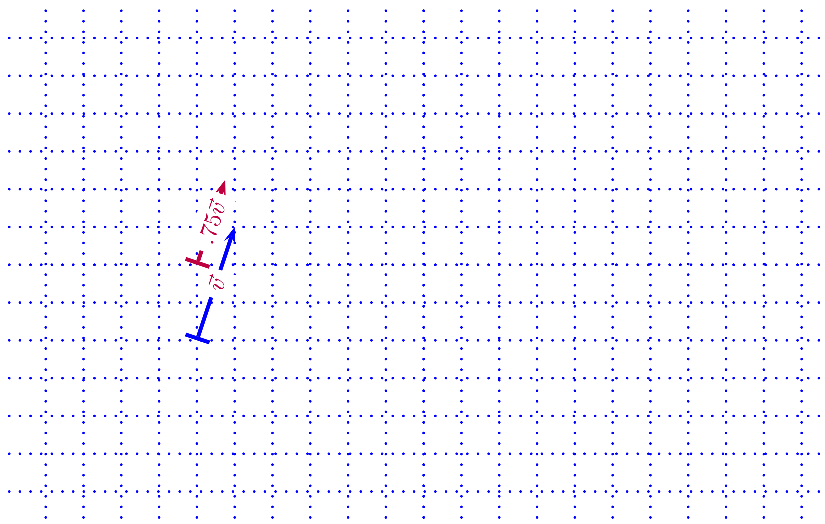


(i) **Vector SCALARS** Suppose $\vec{v} = <1, 3>$compute

$$.75\vec{v}$$

Solution:

$$\begin{aligned} .75\vec{v} &= .75{<}1, 3{>} && \text{(given)} \\ &= {<}.75(1), .75(3){>} && \text{(def of scalar multiplication)} \\ &= {<}0.75, 2.25{>} && \text{(by inspection)} \end{aligned}$$

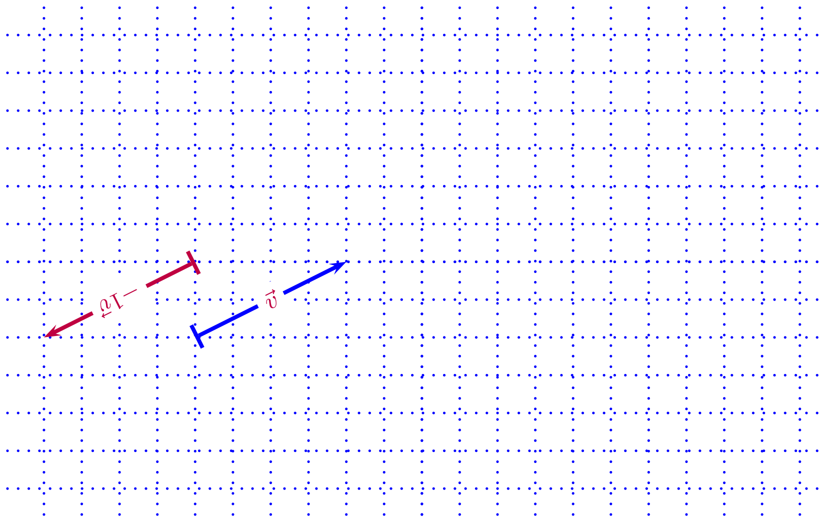


(j) **Vector SCALARS** Suppose $\vec{v} = {<}4, 2{>}$compute

$$-1\vec{v}$$

Solution:

$$\begin{aligned} -1\vec{v} &= -1{<}4, 2{>} && \text{(given)} \\ &= {<}-1(4), -1(2){>} && \text{(def of scalar multiplication)} \\ &= {<}-4, -2{>} && \text{(by inspection)} \end{aligned}$$

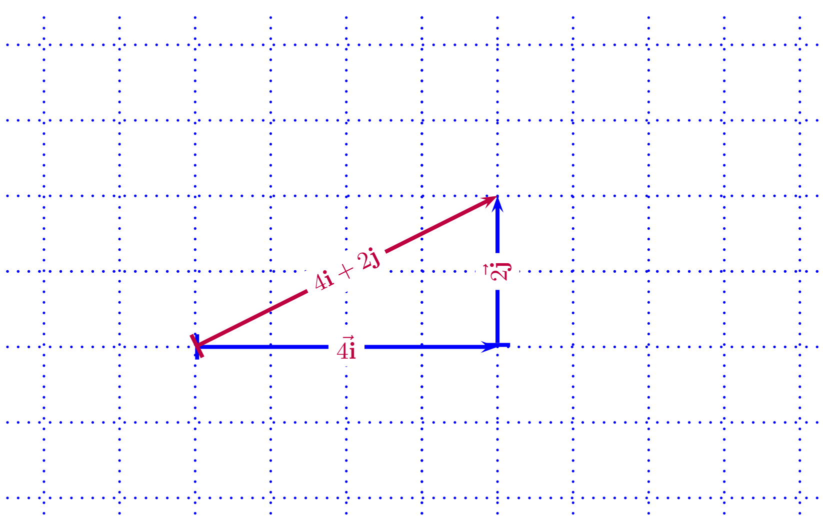


4. **Vector Arithmetic: Famous Vectors** There are two very famous vectors, there are: $\mathbf{i} =< 1, 0 >$ and $\mathbf{j} =< 0, 1 >$.
   (a) **FAMOUS VECTORS i and j** Compute and draw the following vectors

$$4\mathbf{i} + 2\mathbf{j}$$

Solution: let

$$\vec{v} = 4\mathbf{i} + 2\mathbf{j}$$
$$= 4 < 1, 0 > +2 < 0, 1 > \quad \text{(given)}$$
$$=< 4, 0 > + < 0, 2 > \quad \text{(def of scalar multiplication)}$$
$$=< 4, 2 >$$

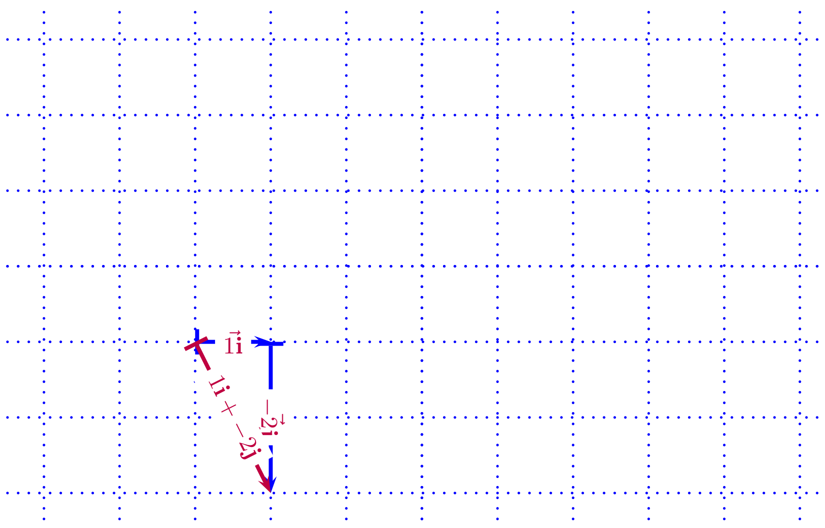


(b) **FAMOUS VECTORS i and j** Compute and draw the following vectors

$$1\mathbf{i} + -2\mathbf{j}$$

Solution: let

$$\begin{aligned} \vec{v} &= 1\mathbf{i} + -2\mathbf{j} \\ &= 1 < 1,0 > + -2 < 0,1 > && \text{(given)} \\ &=< 1,0 > + < 0,-2 > && \text{(def of scalar multiplication)} \\ &=< 1,-2 > \end{aligned}$$

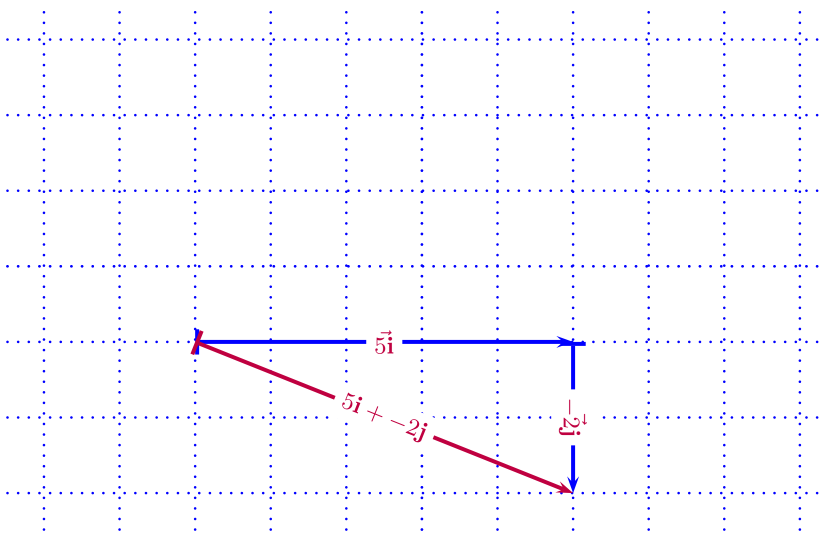


(c) **FAMOUS VECTORS i and j** Compute and draw the following vectors

$$5\mathbf{i} + -2\mathbf{j}$$

Solution: let

$$\begin{aligned} \vec{v} &= 5\mathbf{i} + -2\mathbf{j} \\ &= 5 < 1,0 > + -2 < 0,1 > && \text{(given)} \\ &=< 5,0 > + < 0,-2 > && \text{(def of scalar multiplication)} \\ &=< 5,-2 > \end{aligned}$$

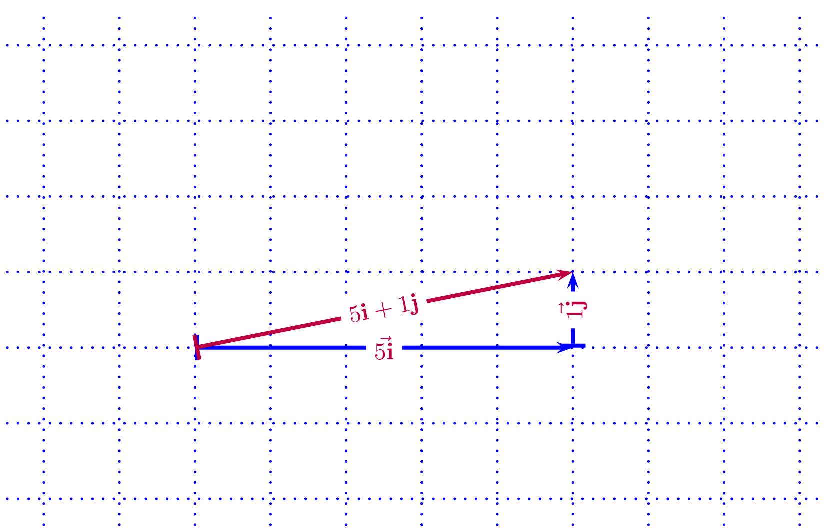


(d) **FAMOUS VECTORS i and j** Compute and draw the following vectors

$$5\mathbf{i} + 1\mathbf{j}$$

Solution: let

$$\begin{aligned}\vec{v} &= 5\mathbf{i} + 1\mathbf{j} \\ &= 5 < 1, 0 > +1 < 0, 1 > && \text{(given)} \\ &=< 5, 0 > + < 0, 1 > && \text{(def of scalar multiplication)} \\ &=< 5, 1 >\end{aligned}$$

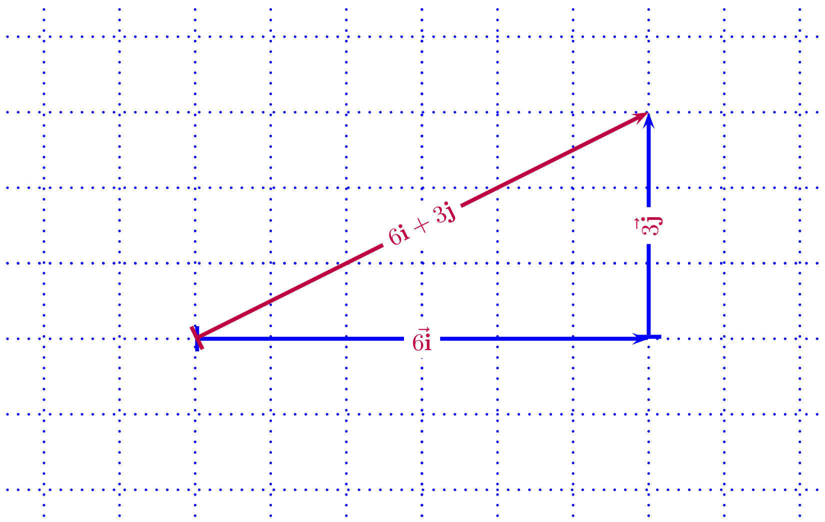


(e) **FAMOUS VECTORS i and j** Compute and draw the following vectors

$$6\mathbf{i} + 3\mathbf{j}$$

Solution: let

$$\begin{aligned}\vec{v} &= 6\mathbf{i} + 3\mathbf{j} \\ &= 6 < 1, 0 > +3 < 0, 1 > && \text{(given)} \\ &=< 6, 0 > + < 0, 3 > && \text{(def of scalar multiplication)} \\ &=< 6, 3 >\end{aligned}$$

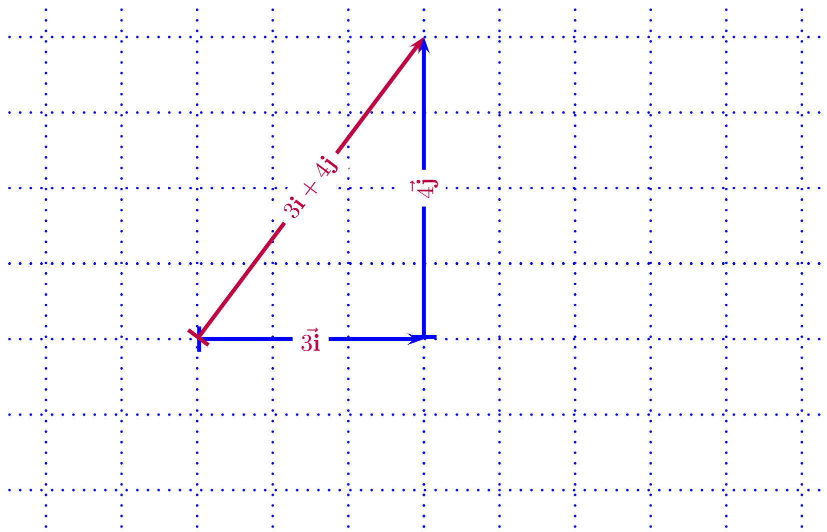


(f) **FAMOUS VECTORS i and j** Compute and draw the following vectors

$$3\mathbf{i} + 4\mathbf{j}$$

Solution: let

$$\begin{aligned}\vec{v} &= 3\mathbf{i} + 4\mathbf{j} \\ &= 3 < 1, 0 > +4 < 0, 1 > && \text{(given)} \\ &=< 3, 0 > + < 0, 4 > && \text{(def of scalar multiplication)} \\ &=< 3, 4 >\end{aligned}$$



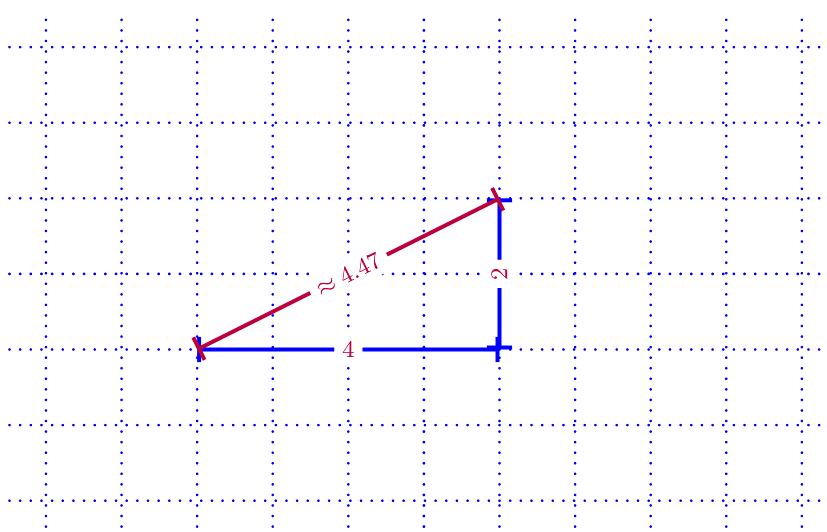
5. **NORM of a VECTOR** Find the norm of the indicated vector.

   (a) Compute:

$$|| < 4, 2 > ||$$

Solution:

$$\begin{aligned} || < 4, 2 > || &= \sqrt{(4)^2 + (2)^2} && \text{(def of norm)} \\ &= \sqrt{20} && \text{(by Inspection)} \\ &\approx 4.47 && \text{(by Calc)} \end{aligned}$$

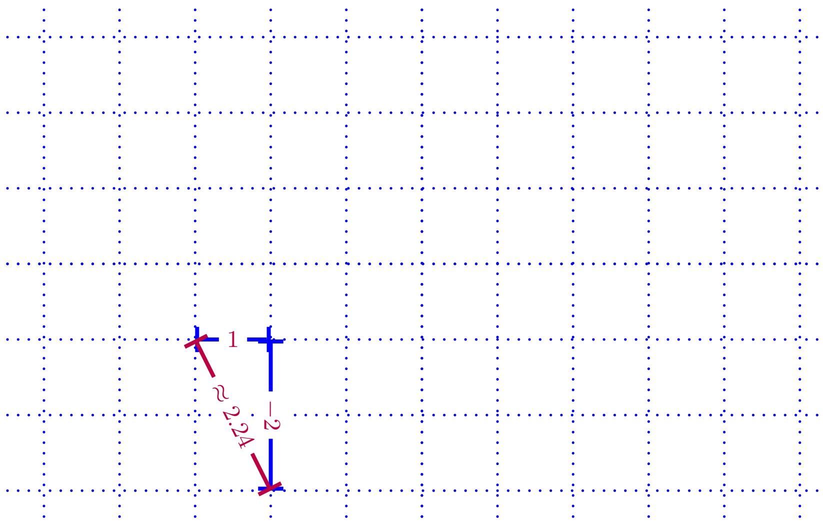


   (b) Compute:

$$|| < 1, -2 > ||$$

Solution:

$$\begin{aligned} || < 1, -2 > || &= \sqrt{(1)^2 + (-2)^2} && \text{(def of norm)} \\ &= \sqrt{5} && \text{(by Inspection)} \\ &\approx 2.24 && \text{(by Calc)} \end{aligned}$$

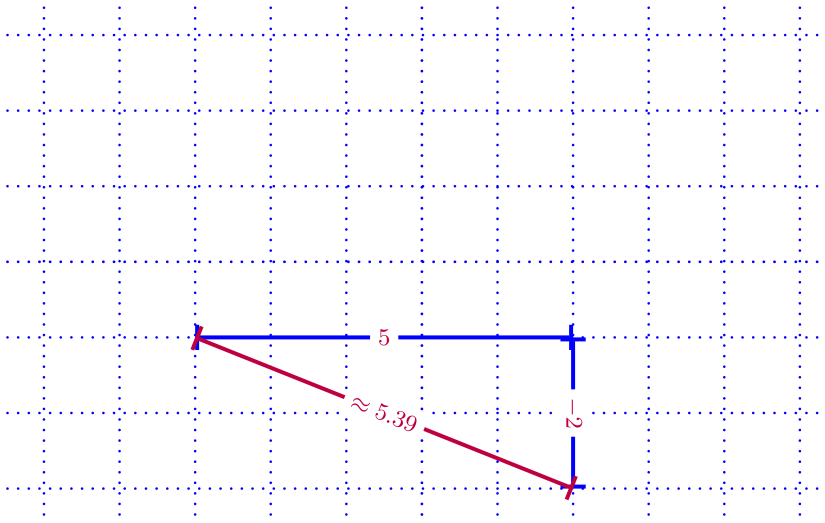


(c) Compute:

$$|| < 5, -2 > ||$$

Solution:

$$\begin{aligned} || < 5, -2 > || &= \sqrt{(5)^2 + (-2)^2} && \text{(def of norm)} \\ &= \sqrt{29} && \text{(by Inspection)} \\ &\approx 5.39 && \text{(by Calc)} \end{aligned}$$

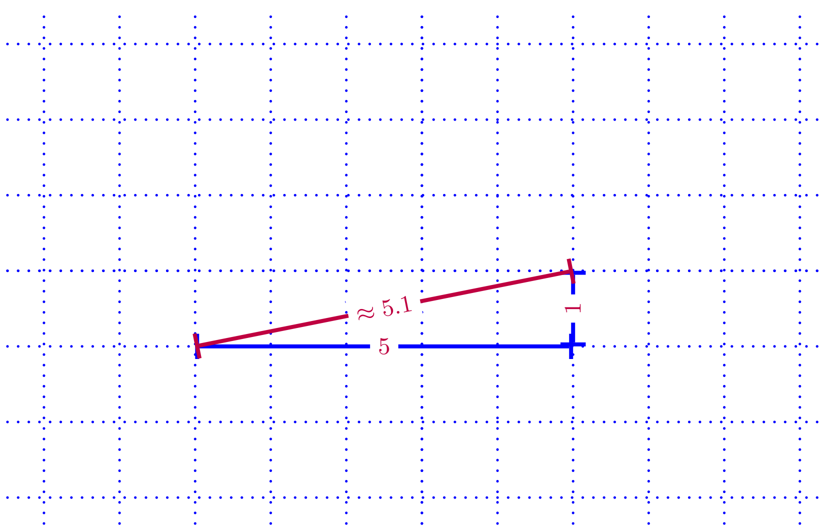


(d) Compute:

$$|| < 5, 1 > ||$$

Solution:

$$|| < 5, 1 > || = \sqrt{(5)^2 + (1)^2} \quad \text{(def of norm)}$$
$$= \sqrt{26} \quad \text{(by Inspection)}$$
$$\approx 5.1 \quad \text{(by Calc)}$$

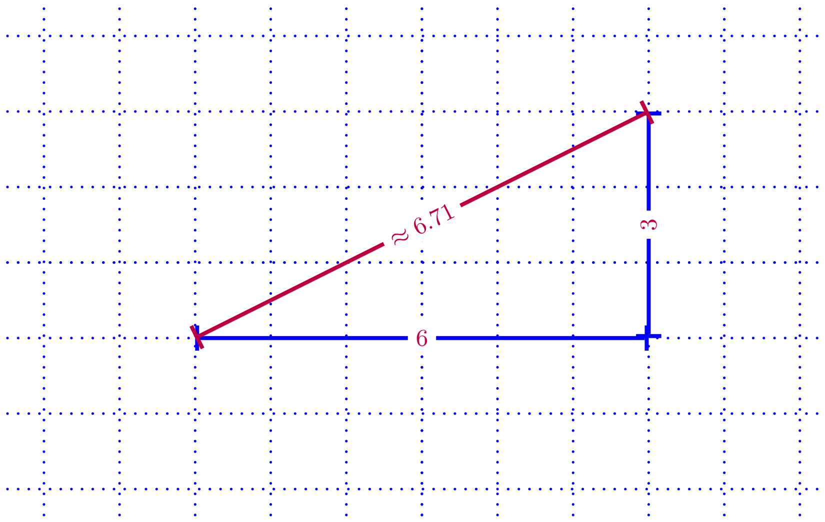


(e) Compute:

$$|| < 6, 3 > ||$$

Solution:

$$\begin{aligned} || < 6,3 > || &= \sqrt{(6)^2 + (3)^2} && \text{(def of norm)} \\ &= \sqrt{45} && \text{(by Inspection)} \\ &\approx 6.71 && \text{(by Calc)} \end{aligned}$$

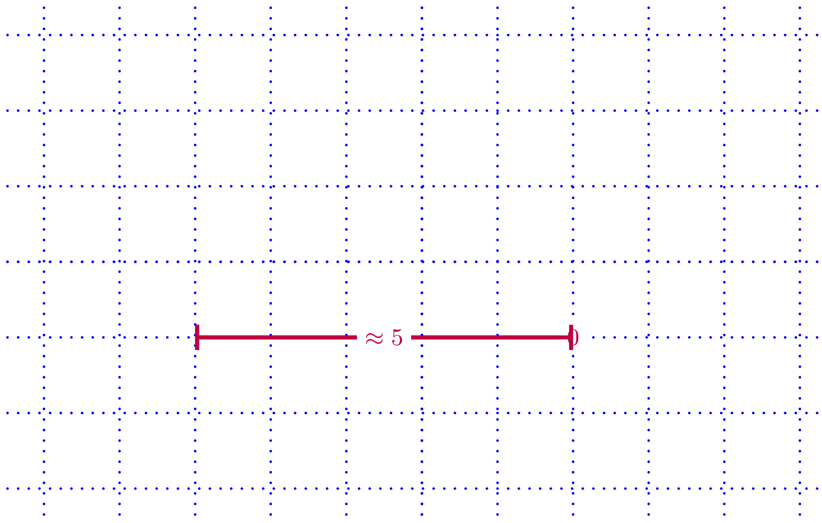


(f) Compute:

$$|| < 5,0 > ||$$

Solution:

$$\begin{aligned} || < 5,0 > || &= \sqrt{(5)^2 + (0)^2} && \text{(def of norm)} \\ &= \sqrt{25} && \text{(by Inspection)} \\ &\approx 5 && \text{(by Calc)} \end{aligned}$$



6. **Which is Bigger?**

(a) Determine which number is larger: $||4\mathbf{i} + 2\mathbf{j}||$ OR $(||4\mathbf{i}|| + ||2\mathbf{j}||)$

Solution:

$$
\begin{aligned}
||4\mathbf{i} + 2\mathbf{j}|| &= || < 4, 2 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\
&= \sqrt{(4)^2 + (2)^2} && \text{(def of norm)} \\
&= \sqrt{20} && \text{(by Inspection)} \\
&\approx 4.47 && \text{(by Calc)}
\end{aligned}
$$

Meanwhile

$$
\begin{aligned}
||4\mathbf{i}|| + ||2\mathbf{j}|| &= || < 4, 0 > || + || < 0, 2 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\
&= 4 + 2 && \text{(def of norm)} \\
&= 6 && \text{(by Inspection)}
\end{aligned}
$$

Therefore, $||4\mathbf{i}|| + ||2\mathbf{j}||$ is larger.

(b) Determine which number is larger: $||1\mathbf{i} + -2\mathbf{j}||$ OR $(||1\mathbf{i}|| + || - 2\mathbf{j}||)$

Solution:

$$
\begin{aligned}
||1\mathbf{i} + -2\mathbf{j}|| &= || < 1, -2 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\
&= \sqrt{(1)^2 + (-2)^2} && \text{(def of norm)} \\
&= \sqrt{5} && \text{(by Inspection)} \\
&\approx 2.24 && \text{(by Calc)}
\end{aligned}
$$

Meanwhile

$$\begin{aligned} ||1\mathbf{i}|| + || - 2\mathbf{j}|| &= || < 1, 0 > || + || < 0, -2 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= 1 + 2 && \text{(def of norm)} \\ &= 3 && \text{(by Inspection)} \end{aligned}$$

Therefore, $||1\mathbf{i}|| + || - 2\mathbf{j}||$ is larger.

(c) Determine which number is larger: $||5\mathbf{i} + -2\mathbf{j}||$ OR $(||5\mathbf{i}|| + || - 2\mathbf{j}||)$

Solution:

$$\begin{aligned} ||5\mathbf{i} + -2\mathbf{j}|| &= || < 5, -2 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= \sqrt{(5)^2 + (-2)^2} && \text{(def of norm)} \\ &= \sqrt{29} && \text{(by Inspection)} \\ &\approx 5.39 && \text{(by Calc)} \end{aligned}$$

Meanwhile

$$\begin{aligned} ||5\mathbf{i}|| + || - 2\mathbf{j}|| &= || < 5, 0 > || + || < 0, -2 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= 5 + 2 && \text{(def of norm)} \\ &= 7 && \text{(by Inspection)} \end{aligned}$$

Therefore, $||5\mathbf{i}|| + || - 2\mathbf{j}||$ is larger.

(d) Determine which number is larger: $||5\mathbf{i} + 1\mathbf{j}||$ OR $(||5\mathbf{i}|| + ||1\mathbf{j}||)$

Solution:

$$\begin{aligned} ||5\mathbf{i} + 1\mathbf{j}|| &= || < 5, 1 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= \sqrt{(5)^2 + (1)^2} && \text{(def of norm)} \\ &= \sqrt{26} && \text{(by Inspection)} \\ &\approx 5.1 && \text{(by Calc)} \end{aligned}$$

Meanwhile

$$\begin{aligned} ||5\mathbf{i}|| + ||1\mathbf{j}|| &= || < 5, 0 > || + || < 0, 1 > || && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= 5 + 1 && \text{(def of norm)} \\ &= 6 && \text{(by Inspection)} \end{aligned}$$

Therefore, $||5\mathbf{i}|| + ||1\mathbf{j}||$ is larger.

(e) Determine which number is larger: $||6\mathbf{i} + 3\mathbf{j}||$ OR $(||6\mathbf{i}|| + ||3\mathbf{j}||)$

Solution:

$$\begin{aligned} ||6\mathbf{i}+3\mathbf{j}|| &= ||<6,3>|| && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= \sqrt{(6)^2+(3)^2} && \text{(def of norm)} \\ &= \sqrt{45} && \text{(by Inspection)} \\ &\approx 6.71 && \text{(by Calc)} \end{aligned}$$

Meanwhile

$$\begin{aligned} ||6\mathbf{i}||+||3\mathbf{j}|| &= ||<6,0>||+||<0,3>|| && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= 6+3 && \text{(def of norm)} \\ &= 9 && \text{(by Inspection)} \end{aligned}$$

Therefore, $||6\mathbf{i}||+||3\mathbf{j}||$ is larger.

(f) Determine which number is larger: $||5\mathbf{i}+0\mathbf{j}||$ OR $(||5\mathbf{i}||+||0\mathbf{j}||)$

Solution:

$$\begin{aligned} ||5\mathbf{i}+0\mathbf{j}|| &= ||<5,0>|| && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= \sqrt{(5)^2+(0)^2} && \text{(def of norm)} \\ &= \sqrt{25} && \text{(by Inspection)} \\ &\approx 5 && \text{(by Calc)} \end{aligned}$$

Meanwhile

$$\begin{aligned} ||5\mathbf{i}||+||0\mathbf{j}|| &= ||<5,0>||+||<0,0>|| && \text{(def of } \mathbf{i} \text{ and } \mathbf{j}) \\ &= 5+0 && \text{(def of norm)} \\ &= 5 && \text{(by Inspection)} \end{aligned}$$

Therefore, $||5\mathbf{i}||+||0\mathbf{j}||$ is larger.

(g) $||\mathbf{u}+\mathbf{v}||$ OR $(||\mathbf{u}||+||\mathbf{v}||)$

Solution: the idea is to study the above pattern and understand that it will always be the case that the sum of the individual norms is larger or equal to the norm of the sum of the vectors. In some sense, this is equivalent to saying that the sum of the lengths of any two sides of a [Euclidean] triangle have to be larger than the size of the length of the the third side of the triangle. This is very famous, it is called the triangle inequality.

7. **Normalize this..** Find the normalized vector for each:

   (a) **NORMALIZE them** Compute and draw the corresponding normalized vector:

$$\vec{v} =< 5, 2 >$$

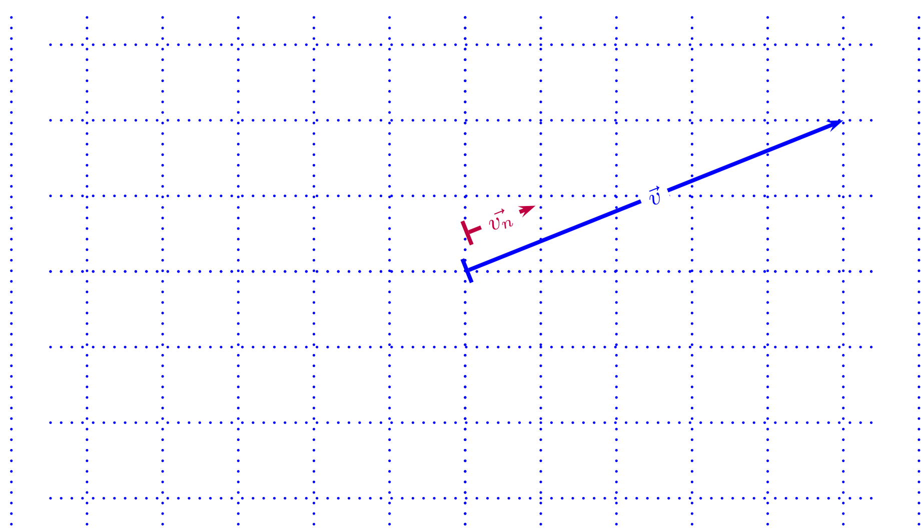
Solution: First we find the norm of the vector:

$$||\vec{v}|| = || < 5, 2 > || = \sqrt{(5)^2 + (2)^2} \quad \text{(def of norm)}$$
$$= \sqrt{29} \quad \text{(by Inspection)}$$
$$\approx 5.39 \quad \text{(by Calc)}$$

Then we scale the original vector $\vec{v}$ by multiplying by $\frac{1}{||\vec{v}||}$. Let us denote the normalized vector $\vec{v}$ as "$\vec{v_n}$"

$$\vec{v_n} = \frac{1}{||\vec{v}||}<5, 2> \quad \text{(given)}$$
$$= \left\langle \frac{5}{||\vec{v}||}, \frac{2}{||\vec{v}||} \right\rangle \quad \text{(def of scalar multiplication)}$$
$$\approx \left\langle \frac{5}{5.39}, \frac{2}{5.39} \right\rangle \quad \text{(approximate)}$$
$$\approx \langle 0.93, 0.37 \rangle \quad \text{(by inspection)}$$

Now, NOTE: $\vec{v}$ and $\vec{v_n}$ have the same direction, BUT, $\vec{v_n}$ has norm "1" as intended. (also keep in mind position does not matter for vectors.. only direction and size, thus nothing should be interpreted from their position, only their size and direction)



(b) **NORMALIZE them** Compute and draw the corresponding normalized vector:

$$\vec{v} = < 5, -2 >$$

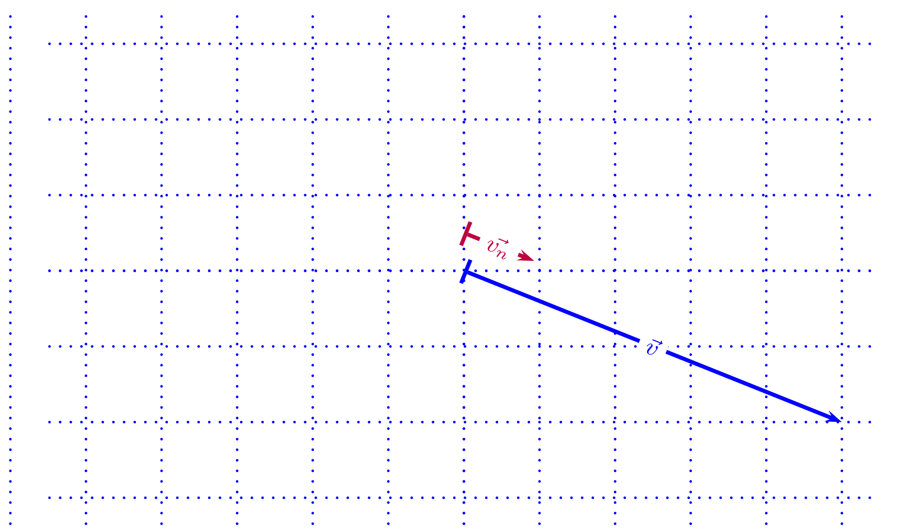
Solution: First we find the norm of the vector:

$$||\vec{v}|| = || < 5, -2 > || = \sqrt{(5)^2 + (-2)^2} \quad \text{(def of norm)}$$
$$= \sqrt{29} \quad \text{(by Inspection)}$$
$$\approx 5.39 \quad \text{(by Calc)}$$

Then we scale the original vector $\vec{v}$ by multiplying by $\frac{1}{||\vec{v}||}$. Let us denote the normalized vector $\vec{v}$ as "$\vec{v_n}$"

$$
\begin{aligned}
\vec{v_n} &= \frac{1}{||\vec{v}||}<5,-2> && \text{(given)} \\
&= \left\langle \frac{5}{||\vec{v}||}, \frac{-2}{||\vec{v}||} \right\rangle && \text{(def of scalar multiplication)} \\
&\approx \left\langle \frac{5}{5.39}, \frac{-2}{5.39} \right\rangle && \text{(approximate)} \\
&\approx \langle 0.93, -0.37 \rangle && \text{(by inspection)}
\end{aligned}
$$

Now, NOTE: $\vec{v}$ and $\vec{v_n}$ have the same direction, BUT, $\vec{v_n}$ has norm "1" as intended. (also keep in mind position does not matter for vectors.. only direction and size, thus nothing should be interpreted from their position, only their size and direction)



(c) **NORMALIZE them** Compute and draw the corresponding normalized vector:

$$\vec{v} =< -3, 4 >$$

Solution: First we find the norm of the vector:

$$
\begin{aligned}
||\vec{v}|| = || < -3, 4 > || &= \sqrt{(-3)^2 + (4)^2} && \text{(def of norm)} \\
&= \sqrt{25} && \text{(by Inspection)} \\
&\approx 5 && \text{(by Calc)}
\end{aligned}
$$

Then we scale the original vector $\vec{v}$ by multiplying by $\frac{1}{||\vec{v}||}$. Let us denote the normalized vector $\vec{v}$ as "$\vec{v_n}$"

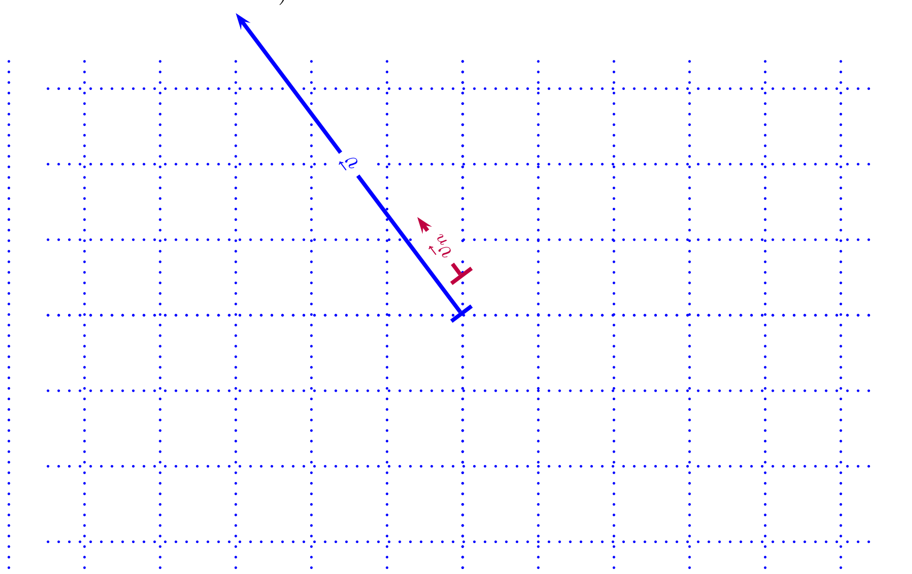
$$\vec{v_n} = \frac{1}{||\vec{v}||} < -3, 4 > \quad \text{(given)}$$

$$= \left\langle \frac{-3}{||\vec{v}||}, \frac{4}{||\vec{v}||} \right\rangle \quad \text{(def of scalar multiplication)}$$

$$\approx \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle \quad \text{(approximate)}$$

$$\approx \langle -0.6, 0.8 \rangle \quad \text{(by inspection)}$$

Now, NOTE: $\vec{v}$ and $\vec{v_n}$ have the same direction, BUT, $\vec{v_n}$ has norm "1" as intended. (also keep in mind position does not matter for vectors.. only direction and size, thus nothing should be interpreted from their position, only their size and direction)



(d) **NORMALIZE them** Compute and draw the corresponding normalized vector:

$$\vec{v} = < -3, -4 >$$

Solution: First we find the norm of the vector:

$$||\vec{v}|| = || < -3, -4 > || = \sqrt{(-3)^2 + (-4)^2} \quad \text{(def of norm)}$$

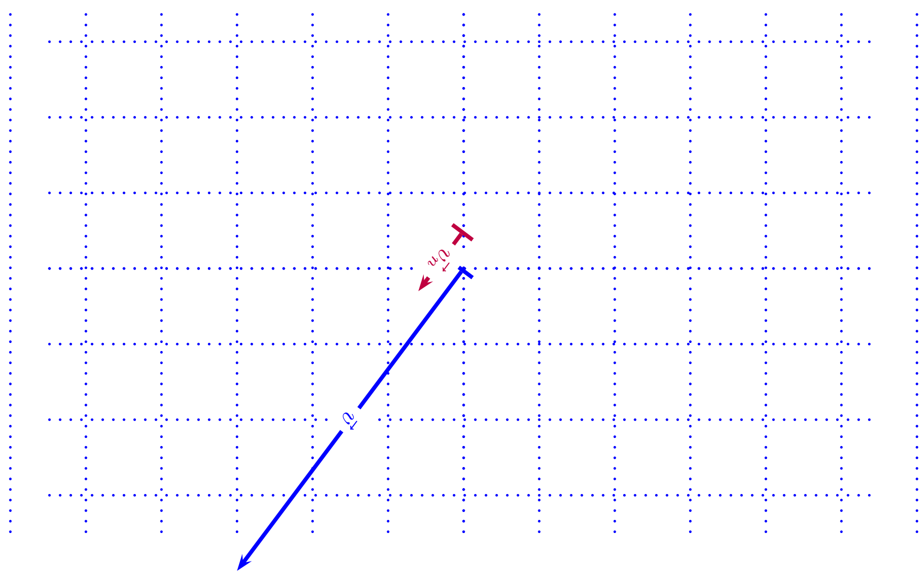
$$= \sqrt{25} \quad \text{(by Inspection)}$$

$$\approx 5 \quad \text{(by Calc)}$$

Then we scale the original vector $\vec{v}$ by multiplying by $\frac{1}{||\vec{v}||}$. Let us denote the normalized vector $\vec{v}$ as "$\vec{v_n}$"

$$\vec{v_n} = \frac{1}{||\vec{v}||} < -3, -4> \quad \text{(given)}$$
$$= \left\langle \frac{-3}{||\vec{v}||}, \frac{-4}{||\vec{v}||} \right\rangle \quad \text{(def of scalar multiplication)}$$
$$\approx \left\langle \frac{-3}{5}, \frac{-4}{5} \right\rangle \quad \text{(approximate)}$$
$$\approx \langle -0.6, -0.8 \rangle \quad \text{(by inspection)}$$

Now, NOTE: $\vec{v}$ and $\vec{v_n}$ have the same direction, BUT, $\vec{v_n}$ has norm "1" as intended. (also keep in mind position does not matter for vectors.. only direction and size, thus nothing should be interpreted from their position, only their size and direction)



(e) **NORMALIZE them** Compute and draw the corresponding normalized vector:

$$\vec{v} = < -3, -1 >$$

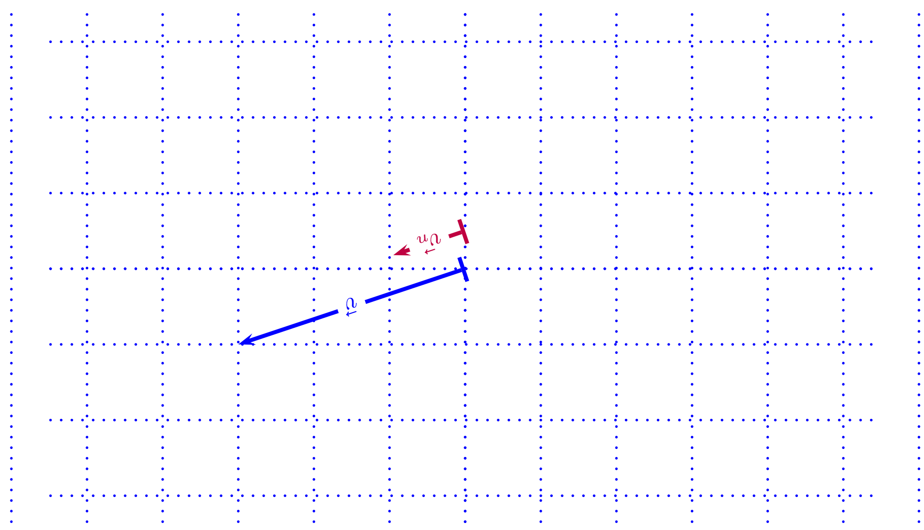
Solution: First we find the norm of the vector:

$$||\vec{v}|| = || < -3, -1 > || = \sqrt{(-3)^2 + (-1)^2} \quad \text{(def of norm)}$$
$$= \sqrt{10} \quad \text{(by Inspection)}$$
$$\approx 3.16 \quad \text{(by Calc)}$$

Then we scale the original vector $\vec{v}$ by multiplying by $\frac{1}{||\vec{v}||}$. Let us denote the normalized vector $\vec{v}$ as "$\vec{v_n}$"

$$\vec{v_n} = \frac{1}{||\vec{v}||}< -3, -1> \quad \text{(given)}$$
$$= \left\langle \frac{-3}{||\vec{v}||}, \frac{-1}{||\vec{v}||} \right\rangle \quad \text{(def of scalar multiplication)}$$
$$\approx \left\langle \frac{-3}{3.16}, \frac{-1}{3.16} \right\rangle \quad \text{(approximate)}$$
$$\approx \langle -0.95, -0.32 \rangle \quad \text{(by inspection)}$$

Now, NOTE: $\vec{v}$ and $\vec{v_n}$ have the same direction, BUT, $\vec{v_n}$ has norm "1" as intended. (also keep in mind position does not matter for vectors.. only direction and size, thus nothing should be interpreted from their position, only their size and direction)



(f) $\vec{v} =< a, b >$ (not both zero..)

Solution:

$$\vec{v_n} = \frac{1}{||\vec{v}||}<a, b> \quad \text{(given)}$$
$$= \left\langle \frac{a}{||\vec{v}||}, \frac{b}{||\vec{v}||} \right\rangle \quad \text{(def of scalar multiplication)}$$
$$= \left\langle \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right\rangle \quad \text{(def of norm)}$$