Vectors: The DOT

What is *the DOT* Product

The dot product is a way of 'multiplying' two vectors together. The result is not a vector but a number. This number is usually referred to as *the DOT product* of the two vectors. Moreover, by itself and at first glance, the dot product appears to be a clumsy nobody. However, spend some time with the DOT and you will find it gives a clean and graceful way to the obtain magnitude of a vector, 'the distance' between two vectors, the angle between two vectors, a simple test to determine if two vectors are perpendicular, and elegant way to project a vector onto another, AND grand generalization to spaces way beyond $\mathbf{R}^2$, way beyond. Having said that, the next task at hand is to actually define the dot product. Let us start with vectors in $\mathbf{R}^2$. For any vector $\mathbf{v} =< a, b >$ and any other vector $\mathbf{u} =< c, d >$, then the dot product is written as $\mathbf{v} \cdot \mathbf{u}$ and is defined as follows:

$$\boxed{< a, b > \cdot < c, d >= ac + bd}$$

some famous *DOT* product properties

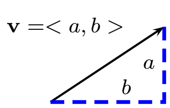
The properties below are precisely what makes the dot product special. It is these properties exactly that allow us to find magnitudes, distances, angles between vectors, all all the other perks that come with being a dot product. This is important to note because, on a different occasion, we may meet a different 'dot product', but so long as it meets these conditions it too will yield a magnitude for vectors, distance, angles, projections, etc... Said differently, anyone who wants to be a dot product has to meet these conditions, and in doing so, will bring with it all the great perks that come with being a dot product.[0]

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	*commutes*
$(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}$	*distributes*
$\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$	*pull constant*
$\mathbf{u} \cdot \mathbf{u} \geq 0$	*self dot pos*
$\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = 0$	*self dot 0*

Magnitude by *the DOT*

Observe what happens when a vector is dotted with itself.

$$\mathbf{v} \cdot \mathbf{v} =< a, b > \cdot < a, b >= a^2 + b^2$$

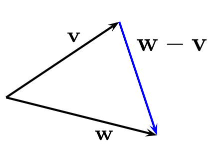


$||\mathbf{v}||^2 = a^2 + b^2$

$$\boxed{||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}}$$

'Distance' by *the DOT*

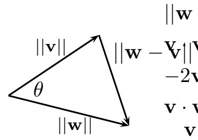
It should be emphasized that the vectors we are working with are defined by their direction and magnitude, NOT by their position. Even without a fixed position we can still define some sort of *distance* between two vectors. This is important because further study of mathematics will lead to great generalizations of the *distance* concept. For the moment, we will be content to note we can measure *a distance* between two vectors in the following way:



$$\boxed{||\mathbf{w} - \mathbf{v}||^2 = (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v})}$$

Angles between vectors by *the DOT*

Here we will use the dot product, dot product properties, and the law of cosines to obtain one of the most famous applications of the dot product. Namely, we will readily obtain the angle between two vectors. Again, it should be emphasized that there are far implications once these ideas are generalized. For example, our world, and our mind may have a little trouble interpreting the angle between two vectors in the 10th dimension, but the 10th dimension is not trouble at all for the dot product. It can effortlessly calculate the angle between two vectors in the 10th dimension or the 1000th dimension.



$$||\mathbf{w}-\mathbf{v}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2||\mathbf{v}||||\mathbf{w}||\cos\theta$$
$$\mathbf{v}\cdot\mathbf{v} - 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} - 2||\mathbf{v}||||\mathbf{w}||\cos\theta$$
$$-2\mathbf{v}\cdot\mathbf{w} = -2||\mathbf{v}||\,||\mathbf{w}||\cos\theta$$
$$\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta$$
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \cos\theta$$

$$\boxed{\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||}}$$

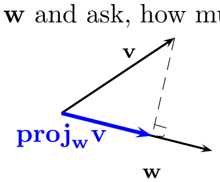
Perpendicular Test *by DOT*

Consider the formula derived above for the $\cos\theta$. Consider interrogating this formula in the case that the vectors were perpendicular. That is, consider what it would tell us if the angle is $90°$. The left side would be $\cos 90° = 0$, this would force the right side to be zero, and this would force the numerator of the right side to be zero. Thus...

$$\boxed{\mathbf{v}\perp\mathbf{w} \iff \mathbf{v}\cdot\mathbf{w} = 0}$$

Projections *by DOT*

We take this oportunity to re-emphasize that vectors are made of two pieces of information, magnitude and direction. We now turn our attention to the direction of a vector. A very insightful way to think about the direction of a vector is to see the direction as a mixture of other directions. For example, the direction of $< 2, 5 >$ is a little to the right and a bit more up. More specifically, 2 to the right, and 5 up. Yet one more way to state it is: the amount of *eastness* in the vector $< 2, 5 >$ is 2, and the amount of *northness* in $< 2, 5 >$ is 5. In this way, we can see the direction of vector, $< 2, 5 >$, as having a little eastness, and some northness. Having said this, we can take any vector $\mathbf{v}$ and some other vector $\mathbf{w}$ and ask, how much $\mathbf{w}$-ness does $\mathbf{v}$ have?



$$\boxed{\mathbf{proj_w v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}}$$

1. **Vector Dot Product**
   (a) Compute the following DOT product:
   $$< 1, 2 > \cdot < 3, 5 >$$
   (b) Compute the following DOT product:
   $$< 3, 5 > \cdot < 3, 5 >$$
   (c) Compute the following DOT product:
   $$< -3, 2 > \cdot < 1, 2 >$$
   (d) Compute the following DOT product:
   $$< 1, 4 > \cdot < -3, -5 >$$
   (e) Compute the following DOT product:
   $$< 2, 3 > \cdot < 3, -3 >$$
   (f) Compute the following DOT product:
   $$< 1, -4 > \cdot < 3, 7 >$$
   (g) $< 2, 1, 3 > \cdot < 1, 2, 1 >$
2. **Vector Dot Product**
   (a) Compute the following DOT product:
   $$(1\mathbf{i} + 2\mathbf{j}) \cdot (0\mathbf{i} + 5\mathbf{j})$$
   (b) Compute the following DOT product:
   $$(3\mathbf{i} + 0\mathbf{j}) \cdot (3\mathbf{i} + 5\mathbf{j})$$
   (c) Compute the following DOT product:
   $$(-3\mathbf{i} + 2\mathbf{j}) \cdot (1\mathbf{i} + 2\mathbf{j})$$
   (d) Compute the following DOT product:
   $$(1\mathbf{i} + 4\mathbf{j}) \cdot (-3\mathbf{i} + -5\mathbf{j})$$
3. **Vector Dot Product** ***to find magnitude***
   (a) Use the dot product to find the magnitude of the following vector:
   $$\vec{v} = < 1, 2 >$$
   (b) Use the dot product to find the magnitude of the following vector:
   $$\vec{v} = < 3, 5 >$$

(c) Use the dot product to find the magnitude of the following vector:

$$\vec{v} =< -3, 2 >$$

(d) Use the dot product to find the magnitude of the following vector:

$$\vec{v} =< 1, 4 >$$

(e) Use the dot product to find the magnitude of the following vector:

$$\vec{v} =< 2, 3 >$$

(f) Use the dot product to find the magnitude of the following vector:

$$\vec{v} =< 1, -4 >$$

(g) Use the dot product to find the magnitude of the following vector:

$$\vec{v} =< 3, -4 >$$

4. **Vector Dot Product *to find 'distance'***

The DOT product provides a way to define the 'distance' between two vectors, $\vec{v}$ and $\vec{w}$. So long as we can define subtraction of the vectors we can define the distance between them as follows:

$$dist(\vec{v}, \vec{w}) = ||\vec{v} - \vec{w}|| = \sqrt{(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})}$$

(a) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< 1, 2 > \qquad \vec{v} =< 3, 5 >$$

(b) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< -3, 2 > \qquad \vec{v} =< 1, 2 >$$

(c) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< 1, 4 > \qquad \vec{v} =< -3, -5 >$$

(d) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< 2, 3 > \qquad \vec{v} =< 3, -3 >$$

(e) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< 1, -4 > \qquad \vec{v} =< 3, 7 >$$

(f) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< 3, -4 > \qquad \vec{v} =< 3, 7 >$$

(g) Use the dot product to find the 'distance' between the indicated vectors.

$$\vec{w} =< 1, -3, 5, 0, 6 > \qquad \vec{v} =< 5, 2, -10, 3, 1 >$$

5. **Vector Dot Product *to find 'angle'* between two vectors**

(a) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <1, 2> \qquad \vec{v} = <3, -5>$$

(b) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <-3, 2> \qquad \vec{v} = <3, 5>$$

(c) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <1, 4> \qquad \vec{v} = <-3, -5>$$

(d) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <2, 3> \qquad \vec{v} = <3, -3>$$

(e) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <3, 7> \qquad \vec{v} = <1, -4>$$

(f) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <3, 9> \qquad \vec{v} = <7, 1>$$

(g) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <-5, 2> \qquad \vec{v} = <2, 5>$$

(h) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <1, 7> \qquad \vec{v} = <7, -1>$$

(i) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <2, 3> \qquad \vec{v} = <3, -2>$$

(j) Use the dot product to find the 'angle' between the indicated vectors.

$$\vec{w} = <1, -3, 5, 0, 6> \qquad \vec{v} = <5, 2, -10, 3, 1>$$

6. **Perpendicular Test by *the DOT*** Find the Dot product for each pair of vectors, then determine if they are perpendicular.

(a) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} = <1, 2> \qquad \vec{v} = <3, -5>$$

(b) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} = <3, 2> \qquad \vec{v} = <2, -3>$$

(c) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} = <1, 4> \qquad \vec{v} = <-3, -5>$$

(d) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} = <2, 3> \qquad \vec{v} = <3, -3>$$

(e) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} =< 3, 6 > \qquad \vec{v} =< 2, -1 >$$

(f) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} =< 3, 12 > \qquad \vec{v} =< -4, 1 >$$

(g) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} =< -5, 2 > \qquad \vec{v} =< 3, 5 >$$

(h) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} =< 1, 7 > \qquad \vec{v} =< 7, -2 >$$

(i) Use the dot product to find test if the indicated vectors are 'perpendicular'.

$$\vec{w} =< 2, 3 > \qquad \vec{v} =< 3, -2 >$$

(j) test to see if perpendicular...

$$\vec{w} =< 1, -3, 5, -2, 6 > \qquad \vec{v} =< 5, 0, -1, 3, 1 >$$

7. **Projections by *the DOT***

(a) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 1, 2 > \qquad \vec{w} =< 5, 0 >$$

(b) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 3, 2 > \qquad \vec{w} =< 5, 0 >$$

(c) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 4, 2 > \qquad \vec{w} =< 5, 0 >$$

(d) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 7, 2 > \qquad \vec{w} =< 5, 0 >$$

(e) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< -3, 2 > \qquad \vec{w} =< 5, 0 >$$

(f) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 1, 2 > \qquad \vec{w} =< 0, 6 >$$

(g) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 3, 2 > \qquad \vec{w} =< 0, 7 >$$

(h) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} =< 4, 2 > \qquad \vec{w} =< 0, 2 >$$

(i) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < 7, 2 > \qquad \vec{w} = < 0, 4 >$$

(j) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < -3, 2 > \qquad \vec{w} = < 0, 3 >$$

(k) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < 1, 2 > \qquad \vec{w} = < 1, 6 >$$

(l) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < 3, 2 > \qquad \vec{w} = < 7, 1 >$$

(m) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < 4, 2 > \qquad \vec{w} = < 4, 2 >$$

(n) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < 7, 2 > \qquad \vec{w} = < 8, 1 >$$

(o) Use the dot product to find the 'projection' of $\vec{v}$ onto $\vec{w}$ then draw such projection.

$$\vec{v} = < 3, 2 > \qquad \vec{w} = < 9, 3 >$$

(p) find projection

$$\vec{v} = < 1, -3, 5, -2, 6 > \qquad \text{onto} \vec{w} = < 5, 0, -1, 3, 1 >$$