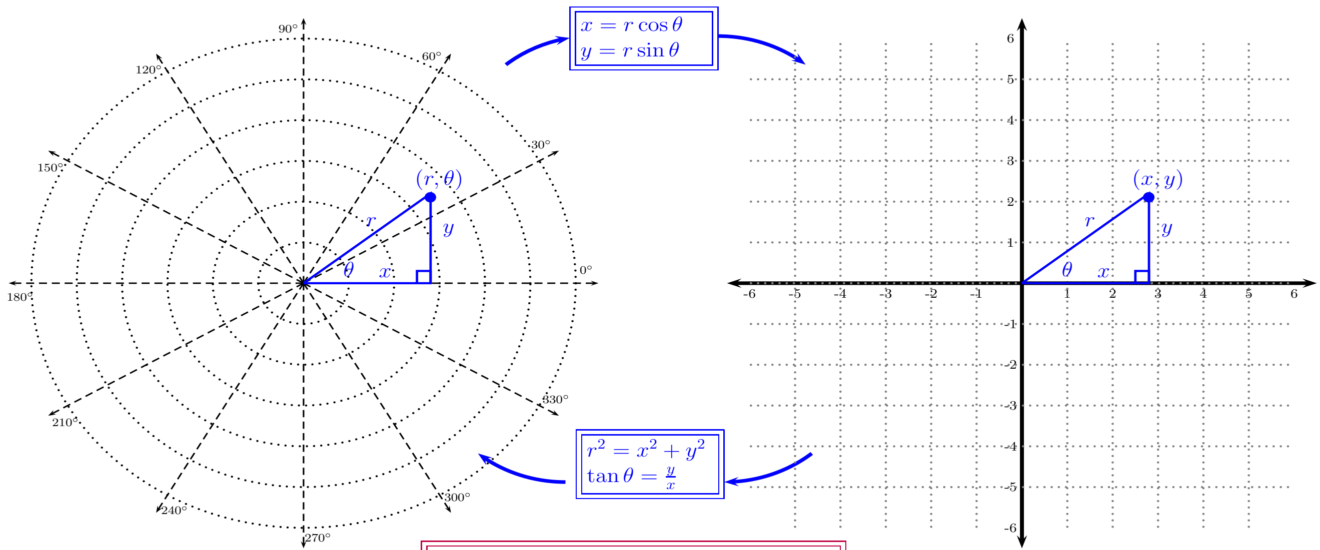
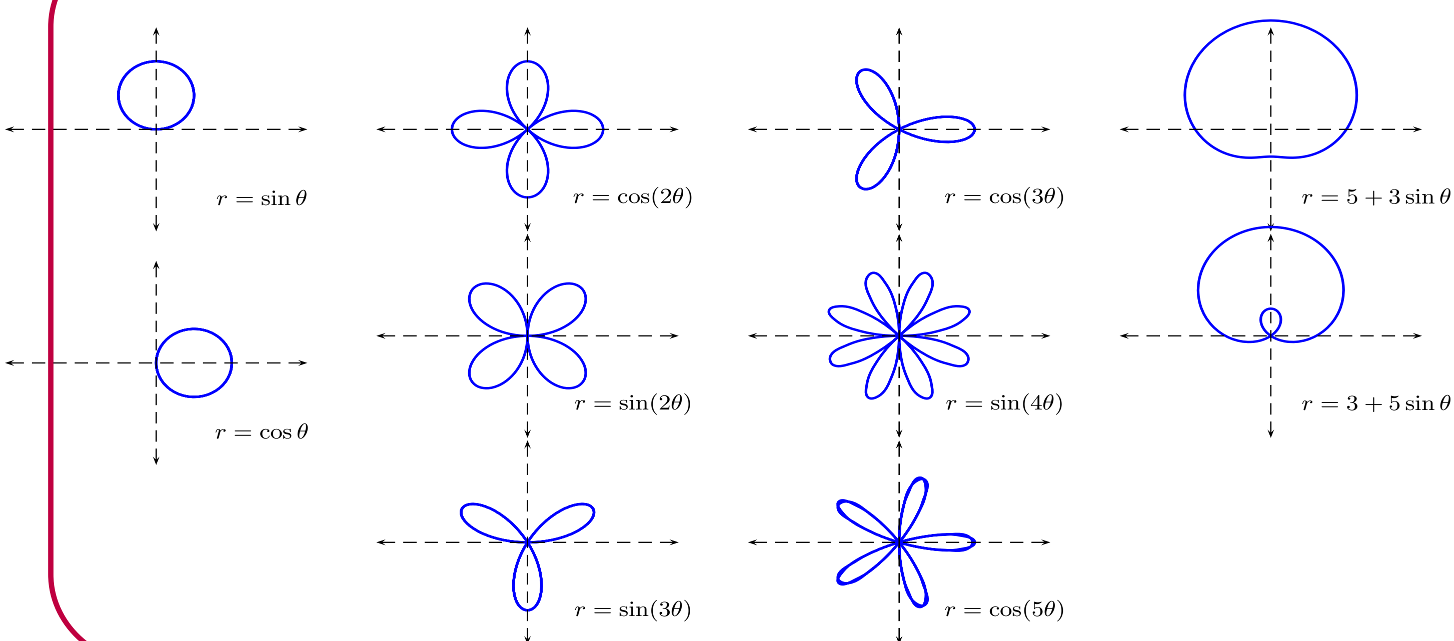
Converting from/to Polar to/from Cartesian

Note, converting from polar to cartesian results in a unique ordered pair, (x, y), while converting to polar results in infinite many possible and correct ordered pairs (r, θ). Also note, when $x = 0$, the quotient $\frac{y}{x}$ is not defined as a real number, in such case, $\theta = 90^\circ + k180^\circ$ or $-90^\circ + k180^\circ$ for $k \in \mathbb{Z}$. Also note, the same method can be used to convert Complex numbers, from cartesian form to euler form.



$$re^{i\theta} = r\cos\theta + ir\sin\theta = x + iy$$

Famous Polar Graphs



Arithmetic ala Euler

Eulers Identity makes a bridge between every complex number, from standard form $a+bi$, to a number of the form $re^{i\theta}$. The great advantage is that multiplying, dividing, taking powers, and finding roots all becomes childs play when numbers are written in Eulers form[0], $re^{i\theta}$.

Multiplying in the World of Euler

$$e^{i30^\circ} \cdot e^{i60^\circ} = e^{i90^\circ}$$

Dividing in the World of Euler

$$\frac{e^{i30^\circ}}{e^{i60^\circ}} = e^{-i30^\circ}$$

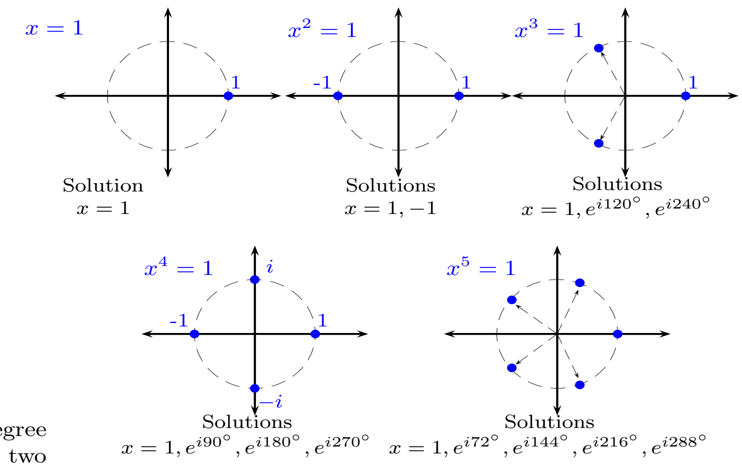
Powers in the World of Euler

$$\left(e^{i30^\circ}\right)^4 = e^{i120^\circ}$$

The Fundamental Theorem of Algebra (F.T.A.)

The Fundamental Theorem of Algebra says a polynomial of degree 1 has exactly one root, a polynomial of degree 2 has exaclty two roots if multiplicity is counted, a polynomial of degree 3 has exaclty three roots if multiplicity is counted, etc. etc... For example, $x=1$ has one solution, $x^2=1$ has two, $x^3=1$ has three, etc.etc...

The Roots of Unity and F.T.A.

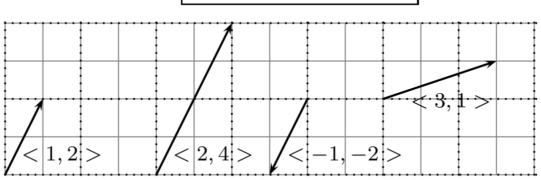


Vector Basics

What are vectors

Each vector is defined by two pieces of information: Direction and Magnitude. Often vectors are described by a picture representation or by ordered pairs which describe the direction and magnitude of the vector. To distinguish from the ordered pairs describing a point, vectors are written using pointy brackets rather than parenthesis. Variables representing vectors are often written in bold or with a hat or arrow over them. Here are a few examples.

Examples of Vectors

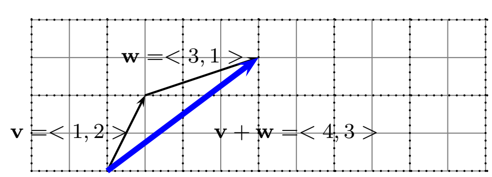


Vector Basics

Adding Vectors

To add two vectors, we simply add the corresponding components together. Using the diagrams to represent vectors we add by placing one vector at the tip of another, tip-to-tail. The vector uniting the starting point to the end point is commonly referred to as the resultant vector or the sum of the vectors. In some sense, the picture of addition of two vectors shows why vectors were invented to begin with. Their roots are in physics, perhaps the most famous example of a vector is a vector representing some Force. because Force is composed to two pieces of information, direction and magnitude, vectors are tailor made to represent such concepts. The direction of the force is represented by the direction of the vector while the magnitude of the force is represented by the magnitude of the vector. With this in mind, adding vectors works just as you may expect if you were adding two forces, tip-to-tail on the picture, component-wise algebraically. For Example:

$$< 1, 2 > + < 3, 1 > = < 4, 3 >$$



Vector Basics

Normalizing of Vectors

To normalize a vector $\mathbf{v}$ refers to producing a new vector $\mathbf{n}_v$ such that this new vector has the same direction as $\mathbf{v}$ BUT magnitude exactly equal to one. Any non-zero vector can be normalized by simply multiplying by a scalar. Such scalar is $1/magnitude_v$. In other words,

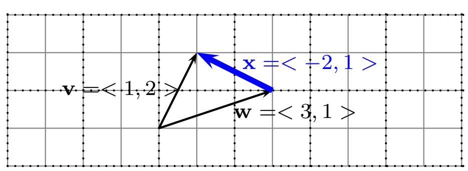
$$\mathbf{n}_v = \frac{1}{||\mathbf{v}||} \cdot \mathbf{v}$$

Subtracting Vectors

The key idea to subtract vectors is to turn the subtraction question into an addition question. For example suppose we want to find $\mathbf{v} - \mathbf{w}$. We can call such vector $\mathbf{x} = \mathbf{v} - \mathbf{w}$. Then we add $\mathbf{w}$ to both sides to get

$$\mathbf{x} + \mathbf{w} = \mathbf{v}$$

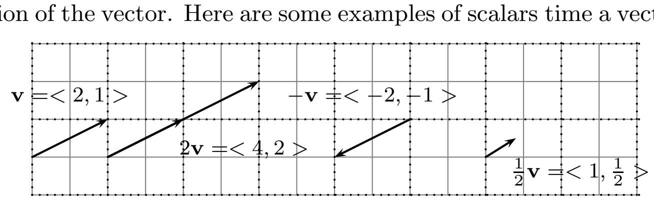
Using the diagrams for each vector we can put $\mathbf{v}$ and $\mathbf{w}$, tail-to tail and solve for x as follows.



Vector Basics

Scalar Multiples of Vectors

Generally, we define the scalar of a vector to be a vector with the same direction (or opposite direction) and possible scaled to a different size. If c is a real number and $\mathbf{v} =< a, b >$ a vector, then $c\mathbf{v} = c < a, b >=< ca, cb >$ is referred to as $\mathbf{v}$ times the scalar c. Note, if the scalar is negative, it reverses the direction of the vector. Here are some examples of scalars time a vector $\mathbf{v}$



Vector DOT Product

What is *the DOT* Product

The dot product is a way of 'multiplying' two vectors together. The result is not a vector but a number. This number is usually referred to as *the DOT product* of the two vectors. Moreover, by itself and at first glance, the dot product appears to be a clumsy nobody. However, spend some time with the DOT and you will find it gives a clean and graceful way to the obtain magnitude of a vector, 'the distance' between two vectors, the angle between two vectors, a simple test to determine if two vectors are perpendicular, and elegant way to project a vector onto another, AND grand generalization to spaces way beyond $\mathbf{R}^2$, way beyond. Let us start with vectors in $\mathbf{R}^2$. For any vector $\mathbf{v} =< a, b >$ and any other vector $\mathbf{u} =< c, d >$, then the dot product is written as $\mathbf{v} \cdot \mathbf{u}$ and is defined as follows:

$$< a, b > \cdot < c, d >= ac + bd$$

Vector DOT Product

some famous *DOT* product properties

The properties below are precisely what makes the dot product special. It is these properties exactly that allow us to find magnitudes, distances, angles between vectors, all all the other perks that come with being a dot product. This is important to note because, on a different occasion, we may meet a different 'dot product', but so long as it meets these conditions it too will yield a magnitude for vectors, distance, angles, projections, etc... Said differently, anyone who wants to be a dot product has to meet these conditions, and in doing so, will bring with it all the great perks that come with being a dot product.

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	*commutes*
$(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}$	*distributes*
$\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$	*pull constant*
$\mathbf{u} \cdot \mathbf{u} \geq 0$	*self dot pos*
$\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = 0$	*self dot 0*

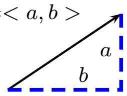
Vector DOT Product

Magnitude by *the DOT*

Observe what happens when a vector is dotted with itself.

$$\mathbf{v}\cdot\mathbf{v} =< a,b > \cdot < a,b >= a^2+b^2$$

$\mathbf{v} =< a,b >$



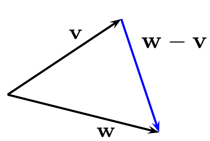
$||\mathbf{v}||^2 = a^2+b^2$

$$\boxed{||\mathbf{v}||^2 = \mathbf{v}\cdot\mathbf{v}}$$

Vector DOT Product

'Distance' by *the DOT*

It should be emphasized that the vectors we are working with are defined by their direction and magnitude, NOT by their position. Even without a fixed position we can still define some sort of *distance* between two vectors. This is important because further study of mathematics will lead to great generalizations of the *distance* concept. For the moment, we will be content to note we can measure *a distance* between two vectors in the following way:

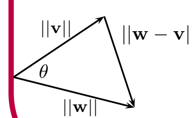


$$\boxed{||\mathbf{w}-\mathbf{v}||^2 = (\mathbf{w}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v})}$$

Vector DOT Product

Angles between vectors by *the DOT*

Here we will use the dot product, dot product properties, and the law of cosines to obtain one of the most famous applications of the dot product. Namely, we will readily obtain the angle between two vectors. Again, it should be emphasized that there are far implications once these ideas are generalized. For example, our world, and our mind may have a little trouble interpreting the angle between two vectors in the 10th dimension, but the 10th dimension is not trouble at all for the dot product. It can effortlessly calculate the angle between two vectors in the 10th dimension or the 1000th dimension.



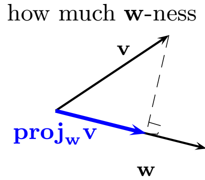
$$||\mathbf{w}-\mathbf{v}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2||\mathbf{v}||||\mathbf{w}||\cos\theta$$
$$\mathbf{v}\cdot\mathbf{v} - 2\mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w} - 2||\mathbf{v}||||\mathbf{w}||\cos\theta$$
$$-2\mathbf{v}\cdot\mathbf{w} = -2||\mathbf{v}||\,||\mathbf{w}||\cos\theta$$
$$\mathbf{v}\cdot\mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta$$
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \cos\theta$$

$$\boxed{\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||}}$$

Vector DOT Product

Projections *by DOT*

We take this oportunity to re-emphasize that vectors are made of two pieces of information, magnitude and direction. We now turn our attention to the direction of a vector. A very insightful way to think about the direction of a vector is to see the direction as a mixture of other directions. For example, the direction of $< 2,5 >$ is a little to the right and a bit more up. More specifically, 2 to the right, and 5 up. Yet one more way to state it is: the amount of *eastness* in the vector $< 2,5 >$ is 2, and the amount of *northness* in $< 2,5 >$ is 5. In this way, we can see the direction of vector, $< 2,5 >$, as having a little eastness, and some northness. Having said this, we can take any vector **v** and some other vector **w** and ask, how much **w**-ness does **v** have?



$$\boxed{\mathbf{proj_w v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}}$$